Monotonicity results for solutions of nonlinear Poisson equation in epigraphs

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12 ième biennale française des mathématiques appliquées et industrielles

05 Juin 2025

joint work with Alberto Farina (LAMFA-UPJV) and Berardino Sciunzi (University of Calabria)

Nonlinear Poisson equation :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{NPE}$$

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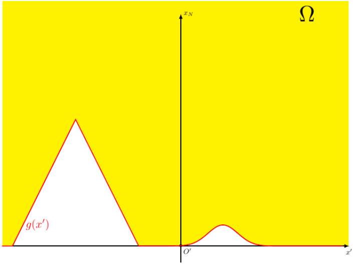
- $f : [0, +\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function, with
$$\boldsymbol{f(0) \geq 0}.$$
- $\Omega \subset \mathbb{R}^N$ is an **epigraph bounded from below**, i.e
$$\Omega := \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$
where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a uniformly continuous function and bounded from below.

Nonlinear Poisson equation :



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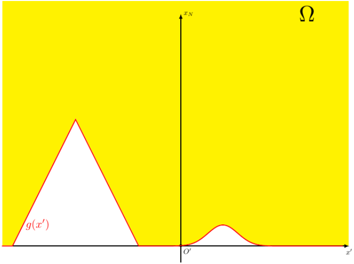

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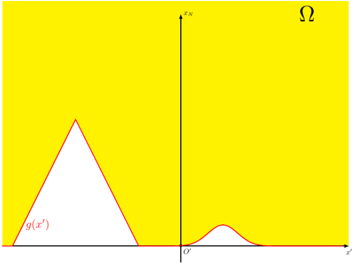

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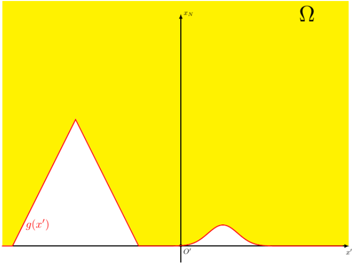

- Aim : Prove the monotonicity of the solution of (NPE) (that is $\frac{\partial u}{\partial x_N} > 0$ in Ω).
- Why :
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  - Liouville-type theorems.

Some classic results

1. If Ω is a ball :
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1. Introduction
   - Presentation of the problem
   - Two relevant results
2. Monotonicity results in an epigraph
   - Results and comments
   - Sketch of the proof : The moving plane method
3. Liouville-type result

- A.FARINA *Some results about semilinear elliptic problems on half-spaces.* Mathematics in Engineering. 709-721.

Theorem (A. Farina (2020))

Assume $N \geq 2$, f a locally Lipschitz function such that $f(0) \geq 0$ and $u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ be a solution of (NPE)*.*
Suppose that

$$\forall t > 0 \quad \exists C(t) > 0, \quad 0 \leq u \leq C(t) \text{ on } \mathbb{R}^{N-1} \times [0, t].$$

Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.

- H. Berestycki, L.A. Caffarelli, L. Nirenberg. *Monotonicity for Elliptic Equations in Unbounded Lipschitz Domains.* Comm. Pure Appl. Math. 1089–1111. 1997

Theorem (Berestycki, Caffarelli, Nirenberg. (1997))

Assume $N \geq 2$, f be an Allen-Cahn type function, Ω be a globally Lipschitz epigraph and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded solution of (NPE).
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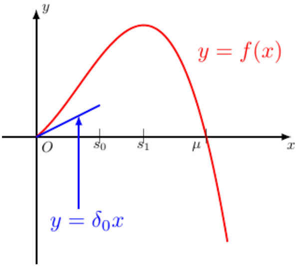

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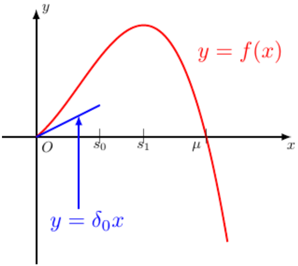

Example :

$$f(x) = x - x^3.$$

Monotonicity results and comments

Theorem (B., Farina, Sciunzi, 2025)

Let Ω *be a globally Lipschitz continuous epigraph bounded from below,* $f \in Lip_{loc}([0, +\infty))$ *with* $f(0) \geq 0$ *and let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *be a solution of* (NPE). *Assume that*

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$$\forall R > 0 \quad \exists C(R) > 0, \quad 0 < u \leq C(R) \text{ on } \Omega \cap \{0 < x_N < R\}.$$

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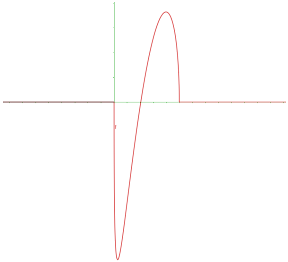
1- If $f \in Lip([0,+\infty))$ and u has at most exponential growth on finite strips, that is, for any $R > 0$,

$$\exists A(R), B(R) > 0\,,\ u(x) \leq Ae^{B|x|} \quad \forall x \in \Omega \cap \{0 < x_N < R\}.$$

then the theorem holds true.

Monotonicity results and comments

2- If f is not locally Lipschitz continuous then the previous theorem does not hold.



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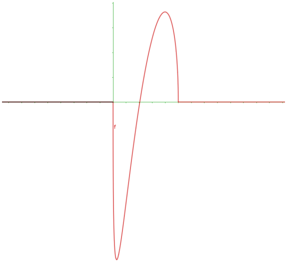

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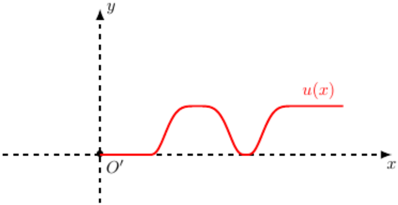
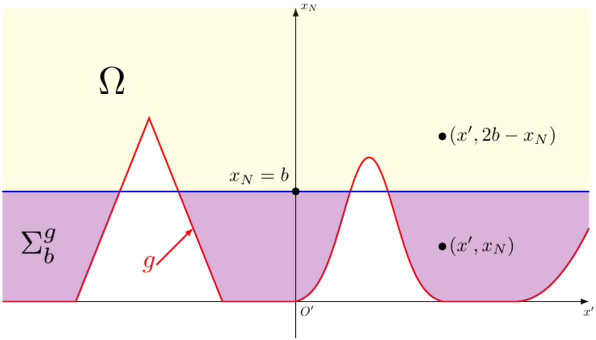


Solution of $-\Delta u = f(u)$

Notations

- $\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N \ : \ g(x') < x_N < b\}$,



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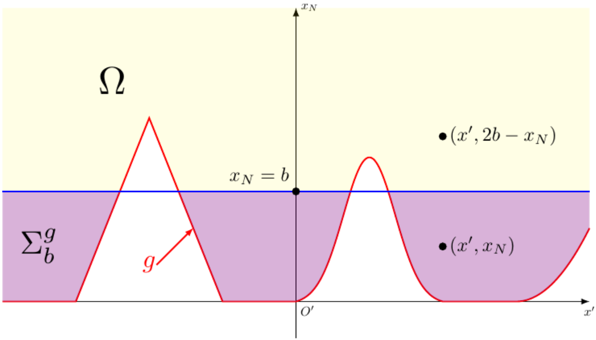

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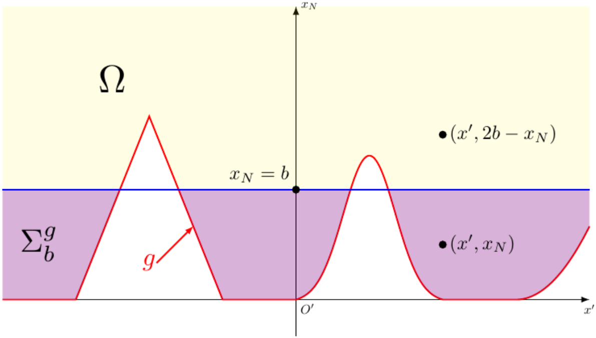

- $\forall x = (x', x_N) \in \Sigma_b^g, \qquad u_b(x) = u(x', 2b - x_N).$

Aim : Prove that

$$\Lambda := \{t > 0 \ : \ u \leqslant u_\theta \text{ in } \Sigma_\theta^g, \ \forall 0 < \theta < t\} = \mathbb{R}_*^+.$$

$\Lambda = \mathbb{R}_*^+$

$$\frac{\partial u}{\partial x_N}(x', b) = \lim_{h \to 0} \frac{u(x', b+h) - u(x', b)}{h},$$

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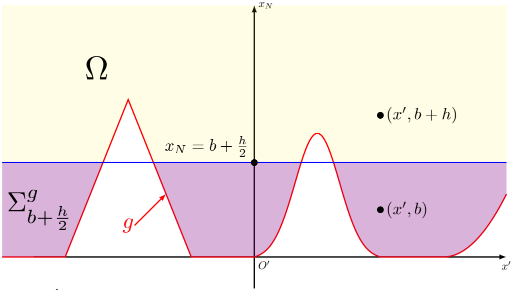

and, since $b + \frac{h}{2} \in \Lambda$ we have

$$u(x) \leq u_{b+\frac{h}{2}}(x) \quad \text{for all } x \in \Sigma^g_{b+\frac{h}{2}}.$$

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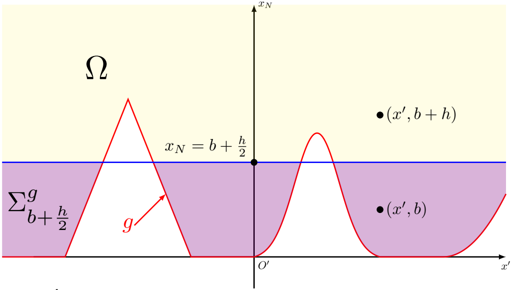

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In particular, as $(x', b) \in \Sigma^g_b$, we get $u(x', b) \leq u(x', b+h)$.

$\Lambda \neq \emptyset$

Theorem (B., Farina, Sciunzi (2025))

Assume $N \geq 2$, $X \subset \mathbb{R}^{N-1} \times [a,b]$ *an open set, let* $M > 0$ *and* $u, v \in C^2(X) \cap C^0(\overline{X})$ *such that*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad X, \\ |u|, |v| \leq M & in \quad X, \\ u \leq v & on \quad \partial X. \end{cases}$$

Then, there exists $\alpha = \alpha(f, M) > 0$ *such that*

$$\sup_{x' \in \mathbb{R}^{N-1}} (\mathcal{L}^1((\{x'\} \times \mathbb{R}e_N) \cap X)) < \alpha \implies u \leq v \text{ in } X.$$

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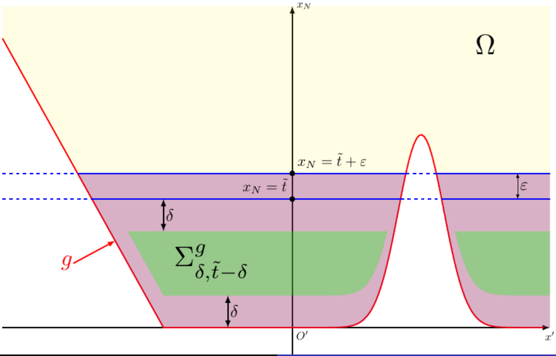
Consequence : $(0, \alpha) \subset \Lambda$.

$\tilde{t} := \sup \Lambda = +\infty$

Proposition ($\tilde{t} < +\infty$)

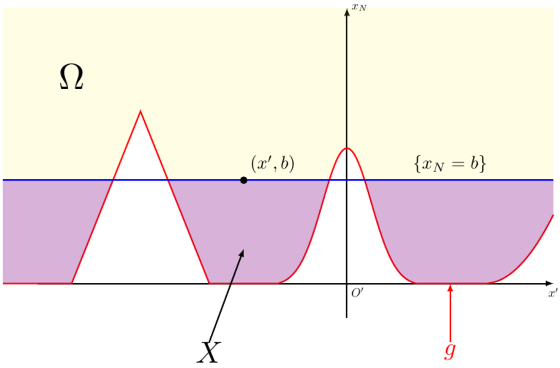
For every $\delta \in (0, \frac{\tilde{t}}{2})$ *there is* $\varepsilon(\delta) > 0$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\tilde{t}+\varepsilon} \quad \text{in} \quad \overline{\Sigma^{g}_{\delta,\tilde{t}-\delta}}.$$



Hopf's Lemma

Let $(x', b) \in \Omega$ and $X \subset \Sigma_b^g$ the connected component such that $(x', b) \in \partial X$.



Hopf's Lemma

Theorem (Hopf's lemma)

Let $w \in C^2(X) \cap C^0(\overline{X})$ *and* $c \geq 0$ *such that*

$$\begin{cases} -\Delta w + cw \geq 0 & \text{in} \quad X, \\ w \geq 0 & \text{in} \quad X, \\ w(x', b) = 0. \end{cases}$$

If $w \not\equiv 0$ *in* X *then*

$$w > 0 \quad \text{in } X \quad \text{and} \quad \frac{\partial w}{\partial x_N}(x', b) < 0.$$

Hopf's Lemma

Applying the Hopf's lemma to $w = u_b - u$ which satisfies

$$w \geq 0 \quad \text{in } \Sigma_b^g, \text{ (since } \Lambda = \mathbb{R}_*^+)$$

and

$$-\Delta w = -\Delta u_b + \Delta u = f(u_b) - f(u) \geq -L_{f,b} w \quad \text{in } \Sigma_b^g.$$

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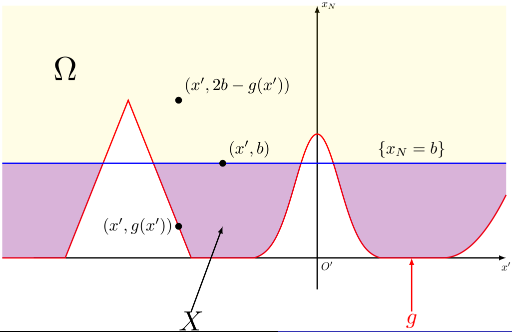
If $w \equiv 0$ in X then $u_b = u$ in X



Hopf's Lemma

Hence $w \not\equiv 0$ in X and as $w = 0$ on $\{x_N = b\}$.

$$0 > \frac{\partial w}{\partial x_N}(x', b) = -2\frac{\partial u}{\partial x_N}(x', b).$$

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Hence $w \not\equiv 0$ in X and as $w = 0$ on $\{x_N = b\}$.

$$0 > \frac{\partial w}{\partial x_N}(x', b) = -2\frac{\partial u}{\partial x_N}(x', b).$$

Therefore

$$\frac{\partial u}{\partial x_N}(x', b) > 0.$$

□

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Assume that $f \in C^1([0, +\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$, then $u \equiv 0$ and $f(0) = 0$.

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Corollary

Let $2 \leq N \leq 11$ and $\Omega \subset \mathbb{R}^N$ be a globally Lipschitz continuous epigraph bounded from below.
If $f \in C^1([0, +\infty))$, satisfies $f(t) > 0$ for $t \geq 0$ then problem (NPE) *does not admit any classical solutions of class $C^2(\Omega) \cap C^0(\overline{\Omega})$.*


Thank You
For Your Attention
MATH

N. Beuvin, A. Farina, B. Sciunzi. *Monotonicity for solutions to semilinear problems in epigraphs.* arXiv :2502.04805v1, 7 Feb 2025.

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