Monotonicity results for solutions of nonlinear Poisson equation in epigraphs

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MAP seminar

26 June 2025, Pau

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LAMFA

1 Introduction
- Presentation of the problem
- Existing works

2 Monotonicity results in an epigraph
- Results and comments
- The moving plane method
- Extensions to merely continuous epigraphs

3 Liouville-type result

Nonlinear Poisson equation :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{NPE}$$

where

- $u \in H^1_{\text{loc}}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ is a distributionnal solution.

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- $f : [0, +\infty) \to \mathbb{R}$ is a locally (or globally) Lipschitz continuous function, with

$$\boldsymbol{f(0) \geq 0}.$$

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$$\boldsymbol{f(0) \geq 0}.$$
- $\Omega \subset \mathbb{R}^N$ is an **epigraph bounded from below**, i.e
$$\Omega := \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$
where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous function and bounded from below.

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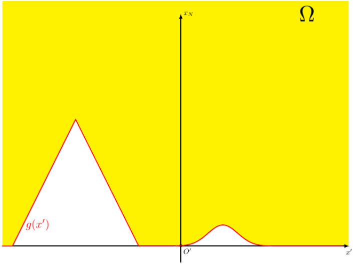

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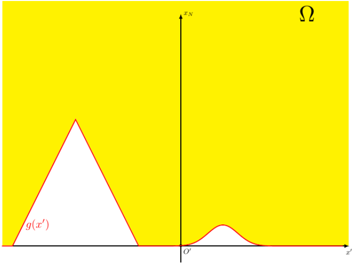

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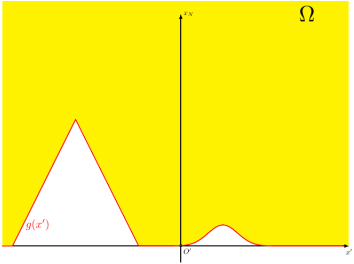

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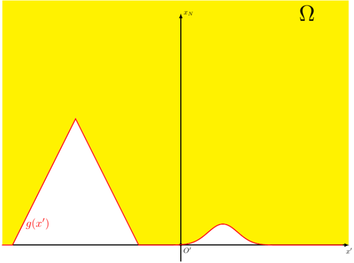

- Aim : Prove the monotonicity of the solution of (NPE) (that is $\frac{\partial u}{\partial x_N} > 0$ in Ω).
- Why :
  - Qualitatives properties (as one-dimensional symmetry),
  - Liouville-type theorems.

- M.J.Esteban-P.L.Lions *Existence and non-existence results for semilinear elliptic problems in unbounded domains.* Proc. Roy. Soc. Edinburgh, 1982, 1-14.

Theorem (Esteban-Lions)

Let $g \in C^1(\mathbb{R}^{N-1})$ such that

$$\lim_{|x'|\to+\infty} g(x') = +\infty,$$

and Ω its epigraph. Let $f \in Lip_{loc}([0,+\infty))$ and $u \in C^2(\overline{\Omega})$ be a classical solution of (NPE).
Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

- H. Berestycki, L.A. Caffarelli, L. Nirenberg. *Monotonicity for Elliptic Equations in Unbounded Lipschitz Domains.* Comm. Pure Appl. Math., 1997, 1089–1111.

Theorem (Berestycki, Caffarelli, Nirenberg. (1997))

Assume $N \geq 2$, f be an Allen-Cahn type function, Ω be a globally Lipschitz epigraph and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded solution of (NPE).
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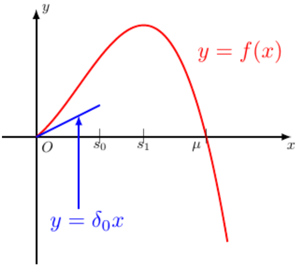

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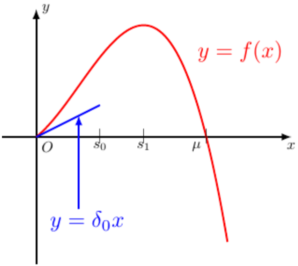

Example :

$$f(x) = x - x^3.$$

- FARINA *Some results about semilinear elliptic problems on half-spaces.* Mathematics in Engineering., 2020, 709-721.

Theorem (A. Farina (2020))

Assume $N \geq 2$, $f \in Lip_{loc}([0,+\infty))$ *such that* $f(0) \geq 0$ *and* $u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ *be a solution of* (NPE).
Suppose that

$$\forall t > 0 \quad \exists C(t) > 0, \quad 0 \leq u \leq C(t) \text{ in } \mathbb{R}^{N-1} \times [0,t].$$

Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ *in* $\mathbb{R}^N_+$.

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Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a uniformly continuous epigraph bounded from below, $f \in Lip([0, +\infty))$ with $f(0) > 0$ and let $u \in H^1_{loc}(\overline{\Omega}) \cap UC(\overline{\Omega})$ be distributionnal solution of (NPE)
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- $H^1_{\text{loc}}(\overline{\Omega}) := \{u : \Omega \mapsto \mathbb{R},\ u \text{ mesurable } :\ u \in H^1(\Omega \cap B(0,R)) \quad \forall R > 0\}$
- $UC(S)$ the set of uniformly continuous functions defined on S.

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- $UC(S)$ the set of uniformly continuous functions defined on S.
- Weierstrass type functions :

$$g_{b,\alpha}(x) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n \pi x), \text{ where } b > 1 \text{ is an integer and } \alpha \in (0,1).$$

The function $g_{b,\alpha}$ is uniformly continuous, bounded and nowhere differentiable.

Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a globally Lipschitz continuous epigraph bounded from below.
Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (NPE) *such that for any $R > 0$, there are positive numbers $A = A(R), B = B(R)$ such that*

$$u(x) \leq Ae^{B|x|} \qquad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Comments

1- If $f \in Lip_{\text{loc}}([0, +\infty))$ and u is bounded on finite strips, that is, for any $R > 0$,

$$\exists C(R) > 0\,,\; u(x) \leq C(R) \quad \forall x \in \Omega \cap \{0 < x_N < R\}.$$

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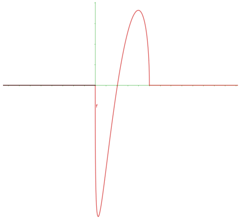

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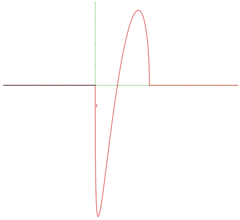

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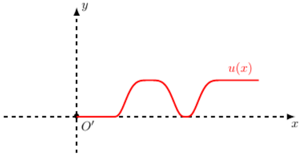
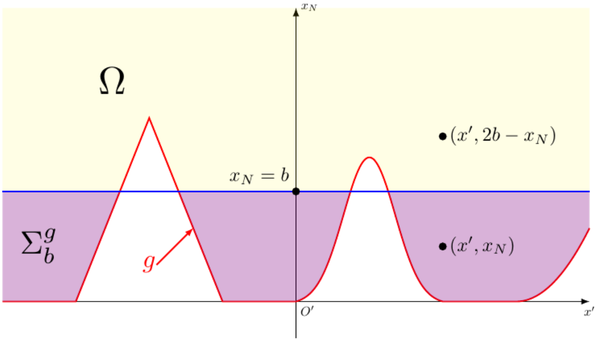


Solution of $-\Delta u = f(u)$

Notations

- $\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N \, : \, g(x') < x_N < b\}$,



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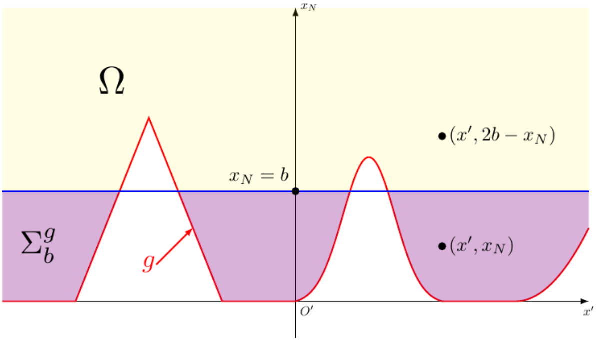

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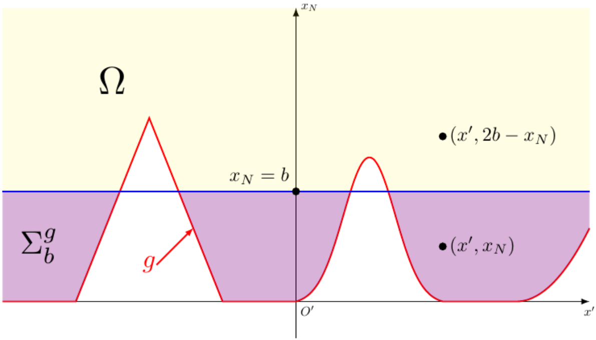

- $\forall x = (x', x_N) \in \Sigma_b^g, \qquad u_b(x) = u(x', 2b - x_N).$

Aim : Prove that

$$\Lambda := \{t > 0 \ : \ u \leqslant u_\theta \ \text{in} \ \Sigma_\theta^g, \ \forall 0 < \theta < t\} = \mathbb{R}_*^+.$$

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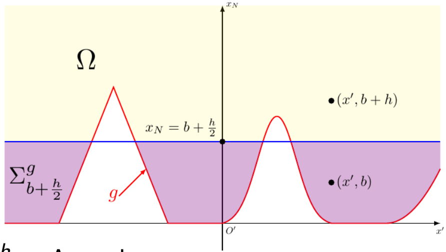

and, since $b + \frac{h}{2} \in \Lambda$ we have

$$u(x) \leq u_{b+\frac{h}{2}}(x) \quad \text{for all } x \in \Sigma^g_{b+\frac{h}{2}}.$$

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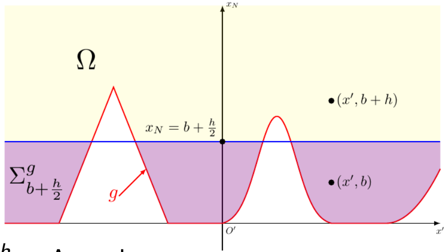

and, since $b + \frac{h}{2} \in \Lambda$ we have

$$u(x) \leq u_{b+\frac{h}{2}}(x) \quad \text{for all } x \in \Sigma^g_{b+\frac{h}{2}}.$$

In particular, as $(x', b) \in \Sigma^g_b$, we get $u(x', b) \leq u(x', b+h)$.

$\Lambda \neq \emptyset$

Definition (Open sets with good section)

An open set $\Omega \subset \mathbb{R}^N$ has a good section in direction e_N if it satisfies the following conditions :

1- For any $R > 0$; we have

$$C_{e_N}(R) = (B'(0', R) \times \mathbb{R}e_N) \cap \Omega \quad \text{is a bounded subset of } \mathbb{R}^N.$$

2-

$$\sup_{x' \in \mathbb{R}^{N-1}} (S_{x'}^{e_N}) := \sup_{x' \in \mathbb{R}^{N-1}} (\mathcal{L}^1((\{x'\} \times \mathbb{R}e_N) \cap \Omega)) < +\infty$$

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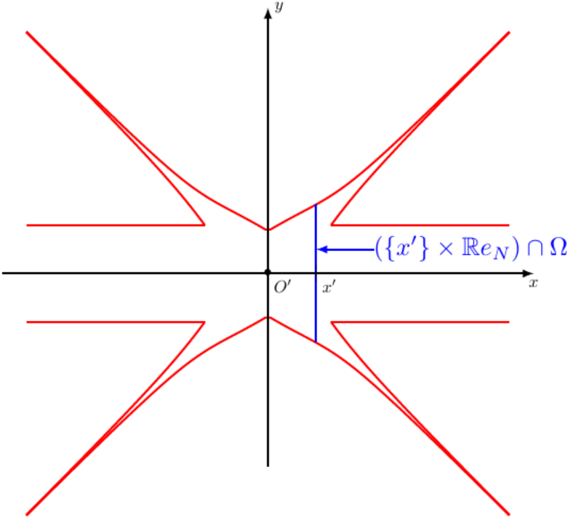

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Theorem (B., Farina, Sciunzi (2025))

Assume $\delta, \gamma \geq 0$, $N \geq 2$ and let Ω be an open subset of $\mathbb{R}^N$ with good section in the direction e_N, such that

$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S_{x'}^{e_N}} |x_N|^{2\delta} e^{2\gamma |x_N|} dx_N \right) < +\infty. \tag{1}$$

Let $f \in Lip(\mathbb{R})$, $a > 0$ and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in} \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^{\delta} e^{\gamma |x|} & \text{in} \quad \Omega, \\ u \leq v & \text{on} \quad \partial\Omega. \end{cases}$$

Then, there exists $\varepsilon = \varepsilon(L_f, \gamma) > 0$ such that

$$S_{e_N}(\Omega) < \varepsilon \implies u \leq v \text{ in } \Omega.$$

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Let $b < 1$ and apply previous theorem with $\Omega = \Sigma_b^g$, $v = u_b$.

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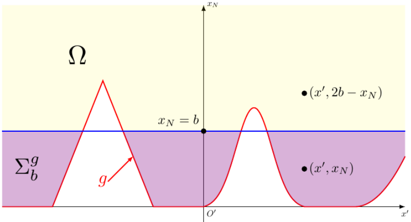
$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S_{x'}^{e_N}} |x_N|^{2\delta} e^{2\gamma |x_N|} dx_N \right) < e^{2\gamma b} b^{2\delta+1}, \text{ for any } \gamma, \delta \geq 0.$$

and

$$\begin{cases} -\Delta u - f(u) = 0 = -\Delta u_b - f(u_b) & \text{in} \quad \mathcal{D}'(\Sigma_g^b), \\ |u|, |u_b| \leq a|x| & \text{in} \quad \Sigma_g^b, \\ u \leq u_b & \text{on} \quad \partial \Sigma_g^b. \end{cases}$$

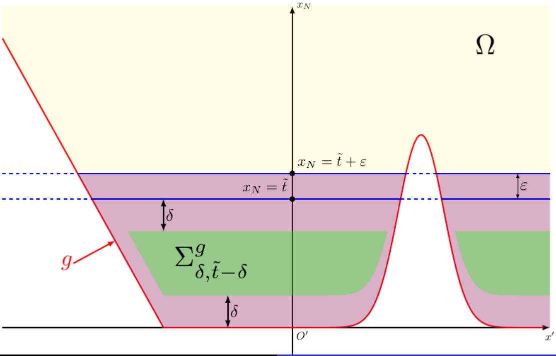


$\tilde{t} := \sup \Lambda = +\infty$

Proposition ($\tilde{t} < +\infty$)

For every $\delta \in (0, \frac{\tilde{t}}{2})$ *there is* $\varepsilon(\delta) > 0$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\tilde{t}+\varepsilon} \quad \text{in} \quad \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}.$$



Proof of Proposition

If the claim were not true, there would exist $\delta \in (0, \frac{\tilde{t}}{2})$ such that

$$\forall\, k \geq 1 \quad \exists\, \varepsilon_k \in \left(0, \frac{1}{k}\right), \exists\, x^k \in \overline{\Sigma^g_{\delta,\tilde{t}-\delta}} \quad : \quad u(x^k) > u_{\tilde{t}+\epsilon_k}(x^k),$$

and so

$$0 \leq g((x^k)') < g((x^k)') + \delta \leq x^k_N \leq \tilde{t} - \delta.$$

thus $x^k_N \to x_\infty$.

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$$\begin{cases} (g_k)_{k\in\mathbb{N}} \text{ is uniformly equicontinuous on } \mathbb{R}^{N-1}, \text{ since } g \in UC(\mathbb{R}^{N-1}), \\ 0 \leq g_k(0') < \tilde{t}. \end{cases}$$

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By Ascoli-Arzelà theorem, there exists $g_\infty \in UC(\mathbb{R}^{N-1})$ such that

$$g_k \to g_\infty \text{ in } C^0_{\text{loc}}(\mathbb{R}^{N-1}).$$

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Then $\tilde{u} \in UC(\mathbb{R}^N)$ (since $u \in UC(\Omega)$) and the sequence $\tilde{u}_k(x) := u((x^k)' + x', x_N)$ satisfies

$$\begin{cases} (\tilde{u}_k)_{k\in\mathbb{N}} \text{ is uniformly equicontinuous on } \mathbb{R}^N, \\ \tilde{u}_k(0', -1) = 0 \,(\text{ since } (0', -1) \in \{x_N < 0\}). \end{cases}$$

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By Ascoli-Arzelà theorem, there exists $\tilde{u}_\infty \in C^0(\mathbb{R}^N)$ such that

$$\tilde{u}_k \to \tilde{u}_\infty \text{ in } C^0_{\text{loc}}(\mathbb{R}^N).$$

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Now, let $u_\infty := \tilde{u}_{\infty_{|\Omega^\infty}}$. Then u_∞ satisfies the following

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- $\tilde{u}_k(0', x_N^k) = u(x^k) > u_{\tilde{t}+\varepsilon_k}(x^k) = \tilde{u}_{k,\tilde{t}+\varepsilon_k}(0', x_N^k)$ so, taking the limit as $k \to +\infty$, we have

$$u_\infty(0', x_N^\infty) \geq u_{\infty,\tilde{t}}(0', x_N^\infty).$$

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By the maximum principle, either

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$$\begin{cases} -\Delta w \geq -L_f w & \text{in} \quad X, \\ w \geq 0 & \text{in} \quad X, \\ w(0', x_\infty) = 0. \end{cases}$$

where $X \subset \Sigma_{\tilde{t}}^{g_\infty}$ is the connected component of $\Sigma_{\tilde{t}}^{g_\infty}$ containing $(0', x_\infty)$.

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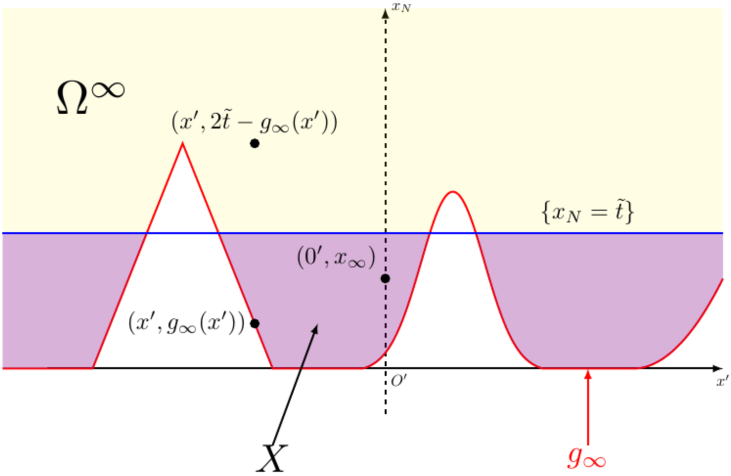
$$w \equiv 0 \text{ in } X, \text{ i.e. } u_\infty \equiv u_{\infty,\tilde{t}} \text{ in } X.$$

Proof of proposition



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$$\left\{ \begin{array}{c} (0,\varepsilon) \in \Lambda, \\ \tilde{t} := \sup \Lambda = +\infty, \end{array} \right. \Rightarrow \Lambda = \mathbb{R}_*^+ \Rightarrow \frac{\partial u}{\partial x_N}(x) \geq 0 \text{ for } x \in \Omega.$$

Now, we will prove that

$$\frac{\partial u}{\partial x_N}(x) > 0 \text{ for any } x \in \Omega.$$

Hopf's Lemma

Theorem (Hopf's lemma)

Let X a smooth domain, $x_0 \in \partial X$ and let ν be the exterior unit normal to X at x_0. Let $w \in C^2(X) \cap C^1(X \cap \{x_0\})$ such that

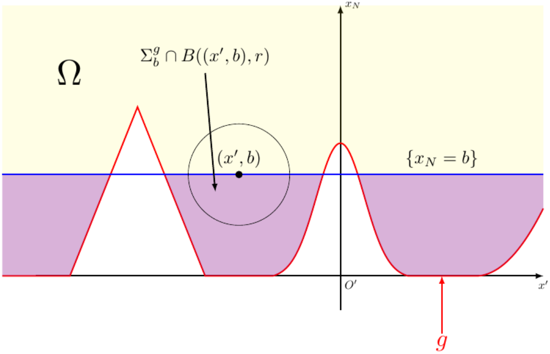
$$\begin{cases} -\Delta w + cw \geq 0 & in \quad X, \\ w > 0 & in \quad X, \\ w(x_0) = 0. \end{cases}$$

Then

$$\frac{\partial w}{\partial \nu}(x_0) < 0.$$

Hopf's Lemma

Let $(x', b) \in \Omega$ and let $r > 0$ such that $B((x', b), r) \subset \Omega$



Hopf's Lemma

Applying the Hopf's lemma with $X = \Sigma_b^g \cap B((x', b), r)$, $x_0 = (x', b)$, $\nu = e_N$ and $w = u_b - u$ which satisfies

$$w > 0 \quad \text{in } X, \text{ (since } \Lambda = \mathbb{R}_*^+ \text{ and by the maximum principle.)}$$

and

$$-\Delta w = -\Delta u_b + \Delta u = f(u_b) - f(u) \geq -L_f w \quad \text{in } \Sigma_b^g.$$

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As $w(x', b) = 0$, we have

$$0 > \frac{\partial w}{\partial x_N}(x', b) = -2\frac{\partial u}{\partial x_N}(x', b).$$

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Therefore

$$\frac{\partial u}{\partial x_N}(x', b) > 0.$$

□

Definition of class $\mathcal{G}$

Definition (Class $\mathcal{G}$)

Assume $N \geq 2$. We say that a continuous function $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ belongs to the class $\mathcal{G}$, if it satisfies the following compactness property

$(\mathcal{P})$ *Any sequence (g_k) of translations of g, which is bounded at some fixed point of $\mathbb{R}^{N-1}$, admits a subsequence converging uniformly on every compact sets of $\mathbb{R}^{N-1}$.*

Example :

- Uniformly continuous functions on $\mathbb{R}^{N-1}$.
- Coercive continuous functions on $\mathbb{R}^{N-1}$.
- Functions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ such that there exists a continuous bijection $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ g \in \mathcal{G}$. As

$$x \to e^x.$$

Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a uniformly continuous epigraph bounded from below, $f \in Lip([0, +\infty))$ with $f(0) > 0$ and let $u \in H^1_{loc}(\overline{\Omega}) \cap UC(\overline{\Omega})$ be distributionnal solution of (NPE)
Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Theorem (B., Farina, Sciunzi, 2025)

Let $N \geq 2$ and let Ω be an epigraph bounded from below and defined by a function $g \in \mathcal{G}$. Assume $f \in Lip([0, +\infty))$ with $f(0) > 0$. If $u \in UC(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ is a distributional solution to (NPE).
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a globally Lipschitz continuous epigraph bounded from below. Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (NPE) with at most exponential growth on finite strips.
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Theorem (B., Farina, Sciunzi, 2025)

Let $N \geq 2$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.
Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (NPE) with at most exponential growth on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Idea of the proof

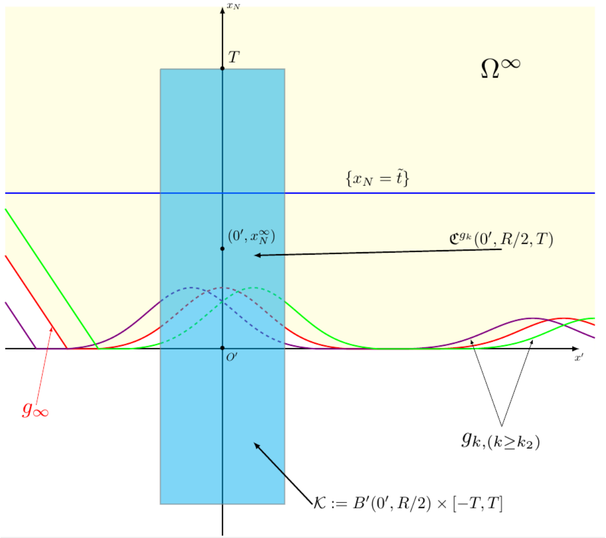
We define in Ω^k,

$$v_k(x) = \frac{u(x' + (x^k)', x_N)}{u(x^k)} = \frac{u_k(x)}{\alpha_k}$$

it satisfies

$$\begin{cases} -\Delta v_k = \frac{f(\alpha_k v_k)}{\alpha_k} := f_k(v_k) & \text{in} \quad \Omega^k, \\ v_k = 0 & \text{in} \quad \partial\Omega^k, \\ v_k(0', x_N^k) = 1. \end{cases}$$

Exterior cone condition implies that $v_k \in C^{0,\alpha}(\overline{\mathfrak{C}^{g_k}(0', R/2, T)})$ and $\|v_k\|_{C^{0,\alpha}(\overline{\mathfrak{C}^{g_k}(0',R/2,T)})} \leq C$ (11 pages)


x_N
T
Ω^∞
$\{x_N = \tilde{t}\}$
$(0', x_N^\infty)$
$\mathfrak{C}^{g_k}(0', R/2, T)$
O'
x'
g_∞
$g_{k,(k \geq k_2)}$
$\mathcal{K} := B'(0', R/2) \times [-T, T]$

We fix $\mathcal{K} := B'(0', R/2) \times [-T, T]$ and

$$\tilde{v}_k(x) = \begin{cases} v_k(x) & \text{if} \quad x \in \overline{\mathfrak{C}^{g_k}(0', R/2, T)}, \\ 0 & \text{if} \quad x \in \mathcal{K} \backslash \overline{\mathfrak{C}^{g_k}(0', R/2, T)}. \end{cases}$$

By Ascoli-Arzela, there exist $v_\infty \in C^{0,\alpha}(\mathcal{K})$ such that $\tilde{v}_k \to v_\infty$ in $C^0(\mathcal{K})$ and it solves.

$$\begin{cases} -\Delta v_\infty = f_\infty(v_\infty) & \text{in} & \mathfrak{C}^{g_\infty}(0', R/2, T), \\ v_\infty \geq 0 & \text{in} & \mathfrak{C}^{g_\infty}(0', R/2, T), \\ v_\infty = 0 & \text{in} & \{x_N = g_\infty(x')\} \cap \mathcal{K}, \\ v_\infty(0', x^\infty) = 1 & \text{and} & v_\infty(0', x^\infty) = v_{\infty, \tilde{t}}(0', x^\infty) \end{cases}$$

By the maximum principle apply v_∞, we have

$$v_\infty > 0 \text{ in } \mathfrak{C}^{g_\infty}(0', R/2, T).$$

By the maximum principle apply to $v_{\infty,\tilde{t}} - v_\infty$ we have

$$v_\infty = v_{\infty,\tilde{t}} \text{ in } \Sigma_{\tilde{t}}^{g_\infty} \cap \mathcal{K}.$$

□

1 Introduction
- Presentation of the problem
- Existing works

2 Monotonicity results in an epigraph
- Results and comments
- The moving plane method
- Extensions to merely continuous epigraphs

3 Liouville-type result

Theorem (B., Farina, Sciunzi (2025))

Let $\Omega \subset \mathbb{R}^N$ be a globally Lipschitz continuous epigraph bounded from below, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded solution to

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Assume that $f \in C^1([0, +\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$, then $u \equiv 0$ and $f(0) = 0$.

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Corollary

Let $2 \leq N \leq 11$ and $\Omega \subset \mathbb{R}^N$ be a globally Lipschitz continuous epigraph bounded from below.
If $f \in C^1([0, +\infty))$, satisfies $f(t) > 0$ for $t \geq 0$ then problem (NPE) *does not admit any classical solutions of class $C^2(\Omega) \cap C^0(\overline{\Omega})$.*

Proof

By the maximum principle, either $u \equiv 0$ and $f(0) = 0$ and the proof is complete, or

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Hence $\frac{\partial u}{\partial x_N} > 0$ in Ω and as $0 \leq u \leq M := \sup_{\Omega} u$, the function

$$v(x') = \lim_{x_N \to \infty} u(x', x_N) \text{ exists and solves}$$

$$\begin{cases} v \in C^2(\mathbb{R}^{N-1}) & \\ -\Delta v = f(v) & \text{in } \mathbb{R}^{N-1}, \\ 0 \leq v \leq M & \text{in } \mathbb{R}^{N-1}, \\ M = \sup_{\mathbb{R}^{N-1}} v. & \end{cases}$$

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and v is stable, ($\int_{\mathbb{R}^{N-1}} f'(v)\phi^2 \leq \int_{\mathbb{R}^{N-1}} |\nabla \phi|^2$).

Proof

Theorem (Dupaigne, Farina, 2022)

Assume that $u \in C^2(\mathbb{R}^p)$ is bounded below and that u is a stable solution of

$$-\Delta u = f(u) \quad \textit{in } \mathbb{R}^p.$$

where $f : \mathbb{R} :\to \mathbb{R}$ is locally Lipschitz and nonnegative. If $1 \leq p \leq 10$, then u must be constant.

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Remarks : This theorem is sharp. Indeed if $p \geq 11$, for $f(u) = u^k$, k sufficiently large, there exists nontrivial positive bounded stable solution to the equation. (see [5]).


Thank You
For Your Attention
MATH

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