Consequences of the monotonicity result on solutions to nonlinear Poisson equation in epigraphs

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PhD students' seminar

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joint work with Alberto Farina (LAMFA-UPJV)

Nonlinear Poisson equation :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{NPE}$$

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- $\Omega \subset \mathbb{R}^N$ is an **epigraph bounded from below**, i.e

$$\Omega := \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a globally Lipschitz continuous function and bounded from below.

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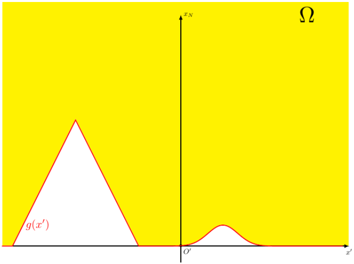

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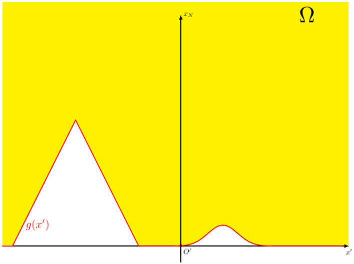

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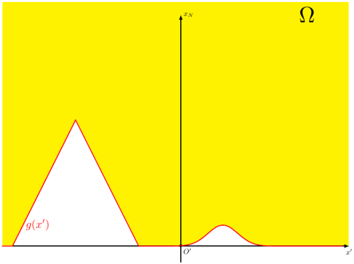

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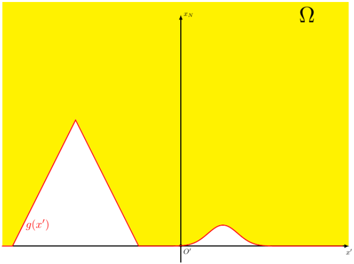

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- Why :
  - Qualitatives properties (as one-dimensional symmetry),
  - Liouville-type theorems.

Monotonicity results in an epigraph

Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a globally Lipschitz continuous epigraph bounded from below.
Assume $f \in Lip([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (NPE) such that for any $R > 0$, there are positive numbers $A = A(R), B = B(R)$ such that

$$u(x) \leq Ae^{B|x|} \qquad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Comments

1- If $f \in Lip_{\mathrm{loc}}([0, +\infty))$ and u is bounded on finite strips, that is, for any $R > 0$,

$$\exists C(R) > 0\,,\ u(x) \leq C(R) \quad \forall x \in \Omega \cap \{0 < x_N < R\}.$$

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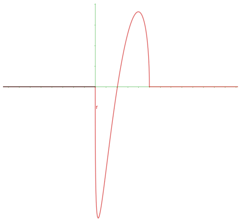

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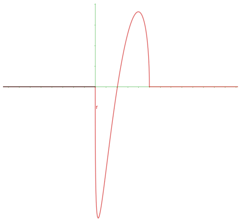

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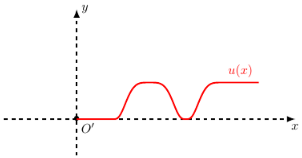

Solution of $-\Delta u = f(u)$

Serrin's overdetermined problem :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \eta} = (\nabla u, \eta) = c & \text{on} \quad \partial\Omega. \end{cases} \qquad \text{(SOP)}$$

where

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Aim : Show one dimensionnal symmetry results that is solutions to SOP depend on only one variable.

On a bounded domain

Theorem (Serrin, 1971)

Let Ω be a bounded domain whose boundary is of class C^2. If there exists a function $u \in C^2(\overline{\Omega})$ satisfying (SOP) *then Ω must be a ball and u is radially symmetric about its center.*

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Physical motivations : Suppose we have a viscous incompressible fluid moving in a straight pipe with a given cross section Ω. The flow velocity u depends only on (x, y) variables and solves

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Then the tangential stress is the same at each point of the boundary (i.e $\frac{\partial u}{\partial \eta} = (\nabla u, \eta) = c$ on $\partial\Omega$.) if and only if Ω is a ball.

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Question : What is the situation when Ω is an unbounded domain ?

On epigraph with Allen-Cahn type nonlinearity

Theorem (Beresticky-Caffarelli-Nirenberg, 1997)

If $g \in C^2$ is a globally Lipschitz-continuous function such that

$$\forall \tau \in \mathbb{R}^{N-1}, \quad \lim_{|x'| \to +\infty} \left(g(x' + \tau) - g(x') \right) = 0.$$

and problem (SOP) *admits a smooth and bounded solution with f an Allen-Cahn type function.*
Then g must be constant (i.e., $\Omega = \{x \in \mathbb{R}^N : x_N > const.\}$ is an upper half-space) and u takes the form $u = u(x_N)$.

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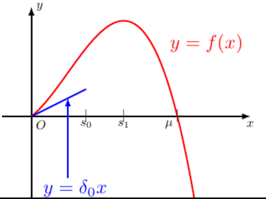

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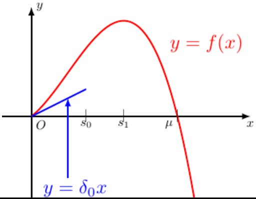

Example :

$$f(x) = x - x^3.$$

Conjecture

Conjecture (Beresticky-Caffarelli-Nirenberg, 1997)

If Ω is a smooth domain with Ω^c connected and that there is a bounded positive solution of (SOP) *for some Lipschitz function f then Ω is either a half space, or a cylinder $\Omega = B_k \times \mathbb{R}^{n-k}$, where B_k is k-dimensional Euclidean ball, or the complement of a ball or a cylinder.*

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In our case, the conjecture becomes

Conjecture

If Ω is a smooth enough epigraph and that there is a bounded positive solution of (SOP) for some Lipschitz function f then Ω is a half space.

Theorem (Del Pino, Pacard, Wei (2014))

Let $f \in C^1([0, +\infty))$ such that

$$f(0) = 0 = f(1), \quad f(s) > 0 \ \forall s \in (0,1), \quad f'(1) < 0.$$

If $N \geq 9$ then there exist an epigraph Ω which is not a half space, such that that the problem

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ 0 < u \leq 1 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \\ \frac{\partial u}{\partial \eta} = (\nabla u, \eta) = c & on \quad \partial\Omega. \end{cases}$$

Remark : This epigraph is not globally Lipschitz continuous.

Theorem (Farina, Valdinoci, 2009)

Let $N = 2, 3$ and f be an Allen-cahn type function.
Let Ω be an open epigraph of $\mathbb{R}^N$ with C^3 and globally Lipschitz boundary.
Suppose that $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (SOP).
Then, we have that $\Omega = \mathbb{R}^N_+$ up to isometry and that there exists $u_0 : (0, +\infty) \to (0, +\infty)$ in such a way that

$$u(x_1, \cdots, x_N) = u_0(x_N) \quad \text{for any } (x_1, \cdots, x_N) \in \Omega.$$

Case $N = 2$

Theorem (B., Farina, 2025)

Let $\Omega \subset \mathbb{R}^2$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3 and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (SOP).
Assume that $f(0) \geq 0$ and one of the following hypotheses holds true :

(H1) $f \in Lip_{loc}([0, +\infty)),$ *and* $\nabla u \in L^\infty(\Omega)$ *;*

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(H1) $f \in Lip_{loc}([0, +\infty))$, *and* $\nabla u \in L^\infty(\Omega)$;

(H2) $\begin{cases} f \in Lip([0,+\infty)), \quad f(t) \leq 0 \quad in \quad (0,+\infty), \\ u(x) = o(|x| \ln^{\frac{1}{2}} |x|), \qquad as \quad |x| \longrightarrow \infty. \end{cases}$

Then, $\Omega = \mathbb{R}^2_+$ up to a vertical translation and there exists $u_0 : [0, +\infty) \to (0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

Important lemma for the proof

Theorem (B., Farina, 2025)

Let $\Omega \subset \mathbb{R}^N$ be a domain of class C^3. Let $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to (SOP) such that

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad \text{as } R \longrightarrow \infty. \tag{1}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{2}$$

then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0, +\infty) \to (0, +\infty)$ strictly increasing such that

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  $\Rightarrow$ The symmetry result stays true when Ω is an half-space.
- We observe that if conditions on ∇u or on u are not satisfy then our symmetry result does not hold. Indeed If we take $u(x) = x_2 e^{x_1}$ then u satisfies

$$\begin{cases} -\Delta u = -u & \text{in} \quad \mathbb{R}^2_+, \\ u > 0 & \text{in} \quad \mathbb{R}^2_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^2_+. \end{cases}$$

Case $N = 3, 4$

Theorem (B., Farina (2025))

Let $N = 3, 4$ and $\Omega \subset \mathbb{R}^N$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3 and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (SOP).
Assume that $f(0) \geq 0$ and

$$(A1)\quad \begin{cases} f \in Lip([0, +\infty)), \quad f(t) \leq 0 \quad in \quad (0, +\infty), \\ u(x) = o(|x|^{\frac{4-N}{2}} \ln^{\frac{1}{2}} |x|), \qquad as \quad |x| \longrightarrow \infty. \end{cases}$$

Then, $\Omega = \mathbb{R}^N_+$ up to a vertical translation and there exists $u_0 : [0, +\infty) \to (0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \quad \forall x \in \mathbb{R}^N_+.$$

We define

$$F(t) = \int_0^t f(s)ds$$

Theorem (B., Farina (2025))

Let $N = 3$ *and* $\Omega \subset \mathbb{R}^N$ *a smooth enough epigraph bounded from below . Let* $f \in C^1([0,+\infty))$ *with* $f(0) \geq 0$ *and* u *a bounded solution to* (SOP). *Suppose that*

$$\sup_{t \in [0, \sup_\Omega u]} F(t) = F(\sup_\Omega u). \tag{3}$$

Then, $\Omega = \mathbb{R}^N_+$ *up to isometry and there exists* $u_0 : [0,+\infty) \to [0,+\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_N) \quad \forall x \in \Omega.$$

Remark :If $N = 2$, then theorem does hold true (without (3)) since

$$0 < u \leq M \Rightarrow \nabla u \text{ is bounded in } \Omega.$$

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- (3) is satisfies if
  1. $f(t) \geq 0$, for any $t \geq 0$.
  2. There exists $\zeta > 0$, such that $f(t) \geq 0$ on $[0, \zeta]$ and $f(t) \leq 0$ on $[\zeta, +\infty)$.
  3. There exists $0 < \zeta_1 < \zeta_2$ such that $f(t) \geq 0$ in $[0, \zeta_1]$, $f(t) \leq 0$ in $[\zeta_1, \zeta_2]$ and $f(t) > 0$ in $(\zeta_2, +\infty)$.

Theorem (B., Farina (2025))

Let $N \geq 2$ and Ω be a smooth enough epigraph bounded from below. Let $f \in C^1(\mathbb{R})$ with $f(0) < 0$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to (SOP) with $c \neq 0$, such that

$$\exists C(R) > 0, \text{ such that } |\nabla u(x)| \leq C(R) \quad \forall x \in \Omega \cap \{x_N \leq R\}.$$

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▪ The proof is based on the moving plane method and $c \neq 0$.

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Assume that one of the following hypotheses holds true :

(H1) $f \in Lip_{loc}([0, +\infty))$, *and* $\nabla u \in L^\infty(\Omega)$;

(H2) $$\begin{cases} f \in Lip([0, +\infty)), & f(t) \leq 0 \quad in \quad (0, +\infty), \\ u(x) = o(|x| \ln^{\frac{1}{2}} |x|), & as \quad |x| \longrightarrow \infty. \end{cases}$$

Then, $\Omega = \mathbb{R}^2_+$ up to a vertical translation and there exists $u_0 : [0, +\infty) \to (0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

Theorem (B., Farina (2025))

Let $N = 3$ *and* $\Omega \subset \mathbb{R}^N$ *a smooth enough epigraph bounded from below . Let* $f \in C^1([0, +\infty))$ *and* u *a bounded solution to* (SOP). *Suppose that*

$$\sup_{t \in [0, \sup_\Omega u]} F(t) = F(\sup_\Omega u).$$

Then, $\Omega = \mathbb{R}^N_+$ *up to isometry and there exists* $u_0 : [0, +\infty) \to [0, +\infty)$ *strictly increasing such that*

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$$u(x) = u_0(x_N) \quad \forall x \in \Omega.$$

- No assumption on the sign of $f(0)$.
- There are always assumptions about u.
- Just for dimensions $N = 2$ or 3.

Theorem (Work in progress)

Let $u \in C^2(\overline{\mathbb{R}^2_+})$ *be a solution of* (NPE) *where* $f \in Lip_{loc}([0,+\infty))$. *Assume that there exists positive constants* η, K, c, σ *such that*

$$f(s) \geq cs^{1+\sigma} - K, \quad s \geq 0 \tag{4}$$

and

$$sf'(s) \geq (1+\eta)f(s) - K, \quad s \geq 0. \tag{5}$$

Then there exists $u_0 : [0,+\infty) \to [0,+\infty)$ *such that*

$$u(x,y) = u_0(y) \quad \textit{for any } (x,y) \in \mathbb{R}^2_+.$$

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- Farina and Sciunzi proved that solutions of (NPE) in $\mathbb{R}^2_+$ are increasing in the x_N direction, independently of the sign of $f(0)$ (see [7]).
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- Theorem can be applied for the Lame-Emden nonlinearity $f(t) = t^p$ $(p > 1)$ or $f(t) = e^t - 2$.

$2 \leq N \leq 9$

Theorem (Work in progress)

Let $2 \leq N \leq 9$ and $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of (NPE) such that u is non-decreasing in the x_N-direction. Assume that $f \in C^1([0, +\infty))$, non-negative and

$$f(t) \geq At - B, \qquad \text{for any } t > 0, \tag{6}$$

for some $A > 0, B \geq 0$.
Then, there exists a function $u_0 : [0, +\infty) \to [0, +\infty)$ bounded, non-decreasing and possibly equal to zero such that,

$$u(x) = u_0(x_N) \quad \text{for any } x \in \mathbb{R}^N_+.$$

Stables solutions

Definition

A solution u of $-\Delta u = f(u)$ in an open set $\Omega \subset \mathbb{R}^N$ is stable if for every $\phi \in C_c^1(\Omega)$ there holds

$$\int_\Omega f'(u)\phi^2 \leq \int_\Omega |\nabla \phi|^2.$$

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Proposition

If $\frac{\partial u}{\partial x_N} > 0$ in Ω then u is stable in Ω.

Theorem (Work in progress)

Let $2 \leq N \leq 9$ and $u \in C^2(\overline{\mathbb{R}^N_+})$ be a stable solution of (NPE), *where $f \in C^1([0, +\infty))$ is a non-negative, non-decreasing and convex function.*
Then, either
- *$u \equiv 0$ and $f(0) = 0$,*

or,
- *there exists $\alpha > 0$ such that*

$$u(x) = \alpha x_N \quad \textit{for any } x \in \mathbb{R}^N_+.$$

Remarks :

- If $u(x) = x_N^2$ then $-\Delta u = -2$ and $f = -2$ is non-decreasing and convex.

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<u>Remarks :</u>

- If $u(x) = x_N^2$ then $-\Delta u = -2$ and $f = -2$ is non-decreasing and convex.
- If $u(x) = \sqrt{x_N}$ then $-\Delta u = \frac{1}{4u^3}$ in $\mathbb{R}^N_+$. (other theory since f is not defined at 0).

Result in $\mathbb{R}^N$

Theorem (Work in progress)

Let $2 \leq N \leq 4$ and $u \in C^2(\mathbb{R}^N)$ a stable solution of $-\Delta u = f(u)$ in $\mathbb{R}^N$. Assume that $f \in Lip_{loc}(\mathbb{R})$ and :

$$\begin{cases} \exists \zeta \in \mathbb{R} \text{ such that } \begin{cases} f(t) \leq 0 & \text{on} & (-\infty, \zeta), \\ f(t) \geq 0 & \text{on} & (\zeta, +\infty). \end{cases} \\ |u(x)| = o(|x|^{\frac{4-N}{2}} \ln^{1/2}|x|) \text{ as } |x| \to +\infty. \end{cases}$$

Then there exists $u_0 : \mathbb{R} \to \mathbb{R}$ such that, up to a rotation,

$$u(x) = u_0(x_N) \quad \text{for any } x \in \mathbb{R}^N.$$

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- The stability is necessary. Indeed, If $u(x) = x_N \sin(x_1)$ then $-\Delta u = u$ in $\mathbb{R}^N$ but u in unstable.


Thank You
For Your Attention
MATH

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