Monotonicity results for solutions of semilinear Poisson equation in epigraphs and applications

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Team EDPA Day

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joint work with Alberto Farina (LAMFA-UPJV) and Berardino Sciunzi (University of Calabria)

Semilinear Poisson equation :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{SPE}$$

where

- $f : [0, +\infty) \to \mathbb{R}$ is a locally or globally Lipschitz-continuous function.

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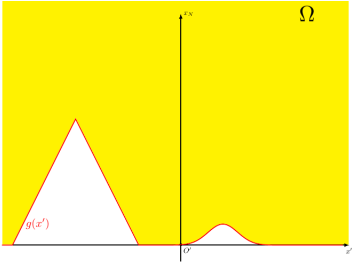
where

- $f : [0, +\infty) \to \mathbb{R}$ is a locally or globally Lipschitz-continuous function.
- $\Omega \subset \mathbb{R}^N$ is an **epigraph bounded from below**, i.e

$$\Omega := \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous function and bounded from below.

Semilinear Poisson equation :



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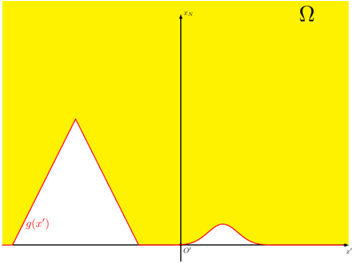

Highlight properties of solutions to the problem (SPE) as :

- monotonicity (i.e, $\frac{\partial u}{\partial x_N} > 0$),
- Classification results.

Motivations

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$$f(u) = u - u^3$$

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Lane-Emden equation :

- Stellar structure in Astrophysics,
- A limit problem, after a blow-up procedure near the boundary (see Gidas-Spruck [a])

a. B. Gidas, J. Spruck. *A Priori Bounds for Positive Solutions of Nonlinear Elliptic Equations.* Commun. in. PDE. 6, 883-901 (1981).

1. Monotonicity results when $f(0) \geq 0$.
   - Monotonicity results in globally Lipschitz-continuous epigraphs.
   - Moving plane method and new comparison principles.
   - Monotonicity results in epigraphs belonging to the class $\mathcal{G}$.
2. Monotonicity results regardless the sign of $f(0)$.
3. Classification results for non-negative solutions to (SPE).

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Existing works

Theorem (Dancer (1992))

In the half-space $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N, x_N > 0\}$, Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution of (SPE). *Assume that $f \in C^1([0, +\infty))$, with $f(0) > 0$ or both $f(0) = 0$ and $f'(0) \geq 0$. Then u is strictly increasing in the x_N-direction, i.e.,*

$$\frac{\partial u}{\partial x_N} > 0 \text{ in } \Omega.$$

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Remark : The need to consider unbounded solutions is well illustrated by following examples :

- $u(x) = \alpha x_N$ with $f \equiv 0$.
- $u(x) = \ln(1 + x_N)$ with $f(t) = e^{-2t}$.

Existing works

Theorem (Berestycki-Caffarelli-Nirenberg (1995))

Assume that $u \in C^2(\overline{\mathbb{R}^N_+})$ is a solution of (SPE) where f is a globally Lipschitz-continuous function with $f(0) \geq 0$. Then, the function u satisfies

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Remark : This Theorem does not applies to function as $f(t) = t - t^3$ (Allen-Cahn equation) or $f(t) = t^p$ (Lane-Emden equation).

Question : Does the conclusion of previous Theorem still holds for unbounded solutions of (SPE) in the case where f is merely locally Lipschitz-continuous ?

Existing works

Theorem (Farina (2020))

Assume $N \geq 2$, $\Omega = \mathbb{R}^N_+$, $f \in Lip_{loc}([0, +\infty))$ *with* $f(0) \geq 0$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution to* (SPE).

$$\forall t > 0,\ \exists C(t) > 0 \text{ such that } 0 < u \leq C(t) \quad \text{in } \mathbb{R}^{N-1} \times [0, t].$$

Then u *is strictly increasing in the* x_N-*direction, i.e,*

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Questions : What happens when the geometry of Ω is more complex ?

Existing works

Theorem (Berestycki, Caffarelli, Nirenberg. (1997))

Assume $N \geq 2$, f be an Allen-Cahn type function, Ω be a globally Lipschitz-continuous epigraph and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a `bounded solution` *of* (SPE).
Then u is monotone, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

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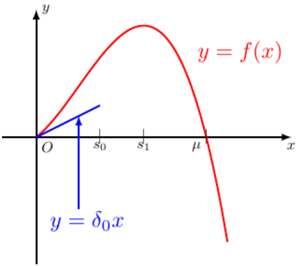

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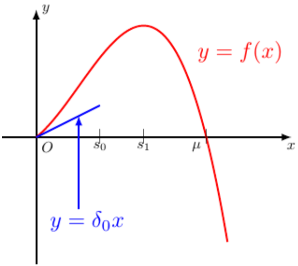

Example :

$$f(u) = u - u^3.$$

Our contributions

Theorem (B.-Farina-Sciunzi (2025))

Let Ω be a globally Lipschitz-continuous epigraph bounded from below. Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ be a distributional solution to (SPE) such that $\forall\, R > 0\ \exists\, A(R), B(R) > 0$, such that,

$$u(x) \leq Ae^{B|x|} \quad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

Then u is strictly increasing in the x_N-direction, i.e.,

$$\frac{\partial u}{\partial x_N} > 0 \ in\ \Omega.$$

$H^1_{loc}(\overline{\Omega}) := \{u : \Omega \to \mathbb{R}, u$ Lebesgue-mesurable : $u \in H^1(\Omega \cap B(0, R)), \forall R > 0\}$.

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$$\sup_{\Omega \cap \{x_N < R\}} u < +\infty.$$

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Remark : This Theorem recovers the Theorem of A.Farina in the case of $\mathbb{R}^N_+$.

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Prove that

$$\Lambda := \{t > 0 \ : \ u \leqslant u_b \text{ in } \Sigma_b^g, \ \forall\, 0 < b < t\} = \mathbb{R}_*^+.$$

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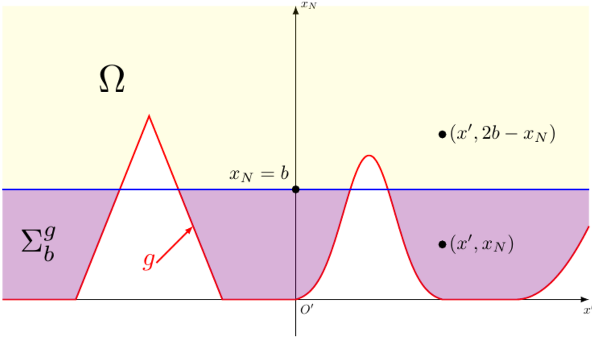

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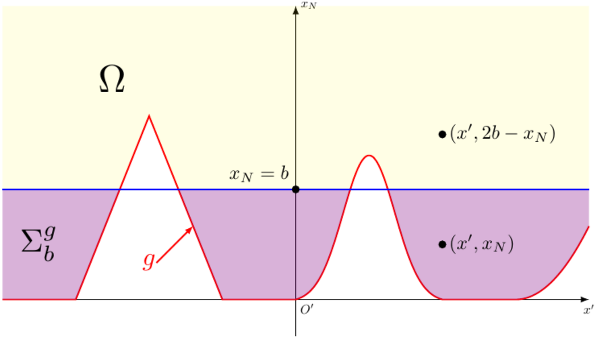

- $\forall x = (x', x_N) \in \Sigma_b^g, \qquad u_b(x) = u(x', 2b - x_N)$.

Step 1 : $\Lambda \neq \emptyset$

- If Σ_b^g is bounded (for instance when Ω is a coercive epigraph) then we can use maximum principles for domains with small volumes. [1]

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- If Σ_b^g is unbounded then $\cdots$

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Step 1 : $\Lambda \neq \emptyset$

Theorem (Farina (2020))

Let $N \geq 2$ and assume that $f \in Lip([0, +\infty))$. Let $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$, with at most polynomial growth at infinity and satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in} \quad \mathbb{R}^{N-1} \times (a, b) \\ u \leq v & \text{on} \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

There exists $\varepsilon = \varepsilon(f) > 0$ such that, for any $(a, b) \subset \mathbb{R}$ with $0 < b - a < \varepsilon$ we have

$$u \leq v \quad \text{in} \quad \mathbb{R}^{N-1} \times (a, b).$$

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Theorem (Berestycki-Caffarelli-Nirenberg (1995))

In $\Omega = \mathbb{R}^{N-1} \times (a, b)$, suppose $w \in W^{2,\alpha}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ is a function satisfying

$$\begin{cases} -\Delta w + c(x, y)w \leq 0 & \text{in} \quad \Omega, \quad \text{with } |c| \leq \gamma, \\ w \leq 0 & \text{in} \quad \partial\Omega, \end{cases}$$

and

$$w \leq Ce^{\mu|x|},$$

for some positive constants γ, C, and μ. There is a constant δ, depending only on N, γ and μ, such that if

$$0 < b - a < \delta,$$

then $$w \leq 0 \text{ in } \Omega.$$

Step 1 : $\Lambda \neq \emptyset$

Definition (Open sets with good section)

An open set $\Omega \subset \mathbb{R}^N$ has a good section in direction e_N if it satisfies the following conditions :

1- For any $R > 0$; we have

$$C_{e_N}(R) = (B'(0', R) \times \mathbb{R}e_N) \cap \Omega \quad \text{is a bounded subset of } \mathbb{R}^N.$$

2-

$$\sup_{x' \in \mathbb{R}^{N-1}} \mathcal{L}^1(S_{x'}^{e_N}) < +\infty,$$

where $S_{x'}^{e_N} := (\{x'\} \times \mathbb{R}e_N) \cap \Omega$.

In the remainder, we fix

$$\mathrm{S}_{e_N}(\Omega) := \sup_{x' \in \mathbb{R}^{N-1}} \mathcal{L}^1(S_{x'}^{e_N}).$$

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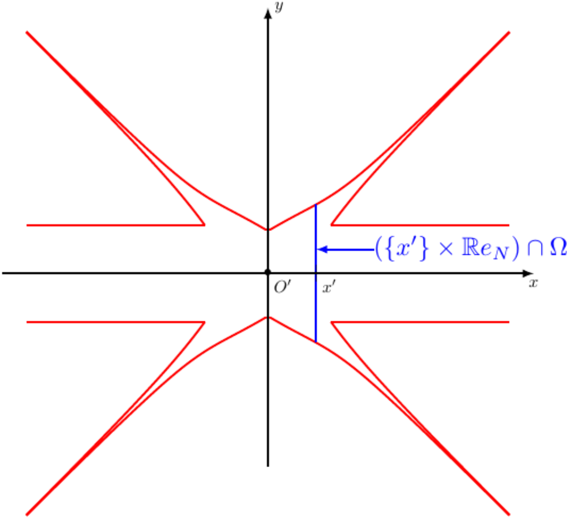

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Theorem (B.-Farina-Sciunzi (2025))

Assume $\delta, \gamma \geq 0$, $N \geq 2$ and let Ω be an open subset of $\mathbb{R}^N$ with good section in the direction e_N. Let $f \in Lip(\mathbb{R})$, $a > 0$ and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^\delta e^{\gamma|x|} & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases}$$

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Assume that Ω satisfies the following property :

$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma|x_N|} dx_N \right) < +\infty.$$

Then, there exists $\varepsilon = \varepsilon(L_f, \gamma) > 0$ such that

$$\mathrm{S}_{e_N}(\Omega) < \varepsilon \implies u \leq v \text{ in } \Omega.$$

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

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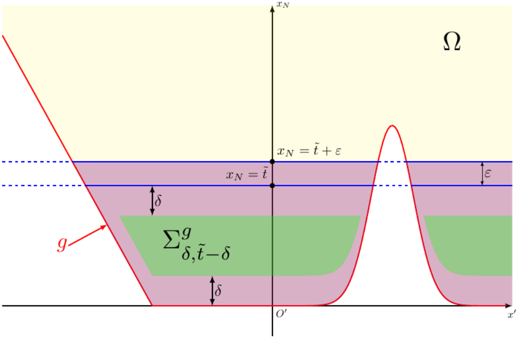
Proposition ($\tilde{t} < +\infty$)

For every $\delta \in (0, \frac{\tilde{t}}{2})$ there is $\varepsilon(\delta) > 0$ such that

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\tilde{t}+\varepsilon} \quad \text{in} \quad \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}.$$



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We are led to consider the following problem :

$$\begin{cases} -\Delta v = f(v) & \text{in} & \mathcal{K}, \\ v \geq 0 & \text{in} & \mathcal{K}, \\ v = 0 & \text{on} & \partial\mathcal{K}, \\ v(x_1) = 1 & \text{where} & x_1 \in \mathcal{K}, \\ v(x_2) = 0 & \text{where} & x_2 \in \mathcal{K}. \end{cases}$$

where $\mathcal{K}$ is a bounded connected set.

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Class $\mathcal{G}$

In the proof, we are led to consider the sequence

$$g_k(x') = g(x' + (x^k)') \text{ where } (g_k(0))_{k\in\mathbb{N}} \text{ is bounded.}$$

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$$g_k \to g_\infty \text{ uniformly on every compact sets of } \mathbb{R}^{N-1}.$$

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Definition (Class $\mathcal{G}$)

Assume $N \geq 2$. We say that a continuous function $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ belongs to the class $\mathcal{G}$, if it satisfies the following compactness property

$(\mathcal{P})$ *Any sequence (g_k) of translations of g, which is bounded at some fixed point of $\mathbb{R}^{N-1}$, admits a subsequence converging uniformly on every compact sets of $\mathbb{R}^{N-1}$.*

Examples of functions belonging in the class $\mathcal{G}$.

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- Uniformly continuous functions on $\mathbb{R}^{N-1}$.
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- Coercive continuous functions on $\mathbb{R}^{N-1}$.
- Functions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ such that there exists a continuous bijection $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ g \in \mathcal{G}$. For instance, if $N = 2$, the function

$$g_1(x) = e^x,$$

belongs in the class $\mathcal{G}$ and is neither uniformly continuous nor coercive.

Monotonicity results

Theorem (B.-Farina-Sciunzi (2025))

Let $N \geq 2$ and let Ω be an epigraph bounded from below defined by a function $g \in \mathcal{G}$ and satisfying a **uniform exterior cone condition.**

Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE).

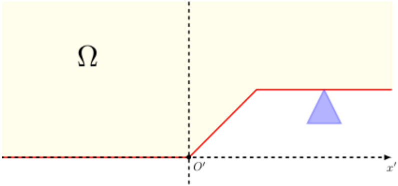
If

- *$f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and u has at most exponential growth on finite strips.*
- *$f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and u is bounded on finite strips.*

Then u is strictly increasing in the x_N-direction, i.e.,

$$\frac{\partial u}{\partial x_N} > 0 \text{ in } \Omega.$$

Uniform exterior cone condition



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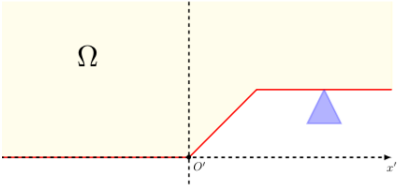

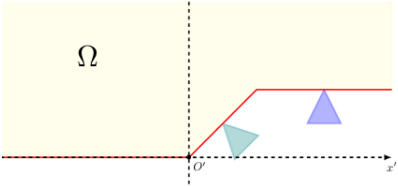

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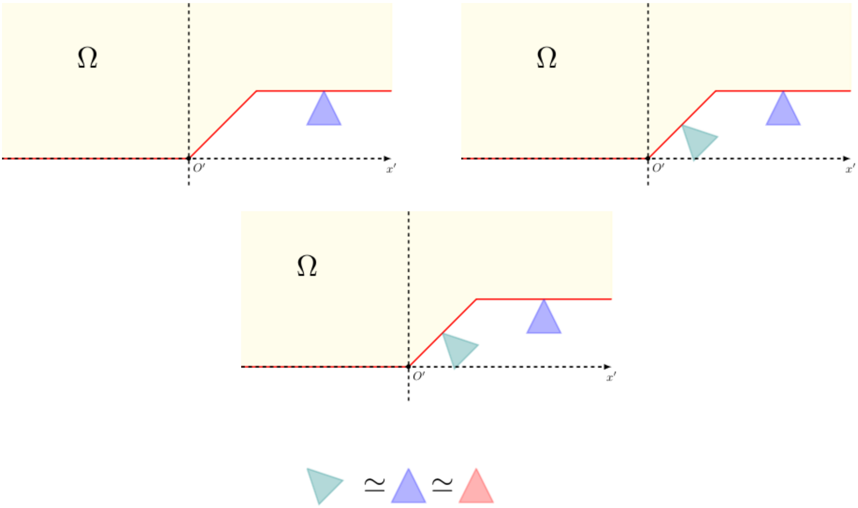

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when $f(0) < 0$

Theorem (Cortázar-Elgueta-Garciá-Melian (2016))

Assume $N \geq 2$. The only classical solution, possibly unbounded, to problem

$$\begin{cases} -\Delta u = u - 1 & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{1}$$

is the function

$$u(x) = 1 - \cos(x_N) \quad \text{for any } x \in \mathbb{R}^N_+.$$

Remark : This Theorem was first demonstrated by A. Farina and B. Sciunzi in dimension 2

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Question : Can we state monotonicity results for solutions of (SPE) defined in epigraphs even if $f(0) < 0$?

Existing works

Theorem (Berestycki-Caffarelli-Nirenberg (1993))

Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution to (SPE) in the half-space, with f a locally Lipschitz-continuous function which satisfies

$$f(\sup u) \leq 0.$$

Then u is strictly increasing in the x_N-direction, i.e, $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$ and u only depends on the variable x_N and $f(\sup u) = 0$.

Existing works

Theorem (Berestycki-Caffarelli-Nirenberg (1993))

Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution to (SPE) *in the half-space, with f a locally Lipschitz-continuous function which satisfies*

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Then u is strictly increasing in the x_N-direction, i.e, $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$ and u only depends on the variable x_N and $f(\sup u) = 0$.

Conjecture

If there is a bounded solution u to (SPE) *with $\Omega = \mathbb{R}^N_+$ then*

$$f(\sup u) = 0.$$

Existing works

Theorem (Berestycki-Caffarelli-Nirenberg (1995))

Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to (SPE) *in $\mathbb{R}^2_+$, with $f \in Lip([0, +\infty))$. Then,*

$$\frac{\partial u}{\partial x_2} > 0 \quad \text{in} \quad \mathbb{R}^2_+.$$

Existing works

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Theorem (Farina-Sciunzi (2014))

Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to (SPE) *in* $\mathbb{R}^2_+$, *with* $f \in Lip_{loc}([0, +\infty))$. *Then,*

$$\frac{\partial u}{\partial x_2} > 0 \quad in \quad \mathbb{R}^2_+.$$

Existing works

Question : What happens when the geometry of Ω is more complex ?

Existing works

Question : What happens when the geometry of Ω is more complex ?

Theorem (Esteban-Lions (1982))

Let $g \in C^1(\mathbb{R}^{N-1})$ such that

$$\lim_{|x'| \to +\infty} g(x') = +\infty,$$

and Ω be its epigraph. Let $f \in Lip_{loc}([0, +\infty))$ and $u \in C^2(\overline{\Omega})$ be a classical solution of (SPE). Then u is strictly increasing in the x_N-direction, i.e,

$$\frac{\partial u}{\partial x_N} > 0 \text{ in } \Omega.$$

Our contributions

Theorem (B.-Farina-Sciunzi (2025))

Let Ω be a coercive continuous epigraph. Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE). Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Remarks :

- There is no assumption concerning the smoothness of Ω and the sign of $f(0)$.

Our contributions

Theorem (B.-Farina-Sciunzi (2025))

Let Ω be a coercive continuous epigraph. Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE). Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Remarks :

- There is no assumption concerning the smoothness of Ω and the sign of $f(0)$.
- We recover Theorem of M.J. Esteban and P.L. Lions.

Our contributions

Theorem (B.-Farina-Sciunzi)

Let Ω be any continuous epigraph bounded from below and let $f \in Lip_{loc}([0, \infty))$ be any non-increasing function. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE) with at most polynomial growth on finite strips.
Then u is strictly increasing, i.e.,

$$\frac{\partial u}{\partial x_N} > 0 \text{ in } \Omega$$

.

The assumption f locally or globally Lipschitz-continuous is sharp.

The assumption f locally or globally Lipschitz-continuous is sharp. The bounded function

$$u(x) = \begin{cases} 1 - (x_N - 1)^4 & \text{if} \quad 0 \leq x_N \leq 1, \\ 1 & \text{if} \quad x_N > 1, \end{cases}$$

is a classical C^2 solution to (SPE), where f is given by the following function

$$f(t) = \begin{cases} 12 & \text{if} \quad t < 0, \\ 12\sqrt{1-t} & \text{if} \quad 0 \leq t \leq 1, \\ 0 & \text{if} \quad t > 1. \end{cases}$$

1. Monotonicity results when $f(0) \geq 0$.
   - Monotonicity results in globally Lipschitz-continuous epigraphs.
   - Moving plane method and new comparison principles.
   - Monotonicity results in epigraphs belonging to the class $\mathcal{G}$.

2. Monotonicity results regardless the sign of $f(0)$.

3. Classification results for non-negative solutions to (SPE).

Existing results

Theorem (Farina (2007))

Let $0 < \alpha < 1$ and let Ω be a $C^{2,\alpha}$ coercive epigraph. Let $u \in C^2(\overline{\Omega})$ be a bounded solution to

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

with

$$\begin{cases} 1 < p < +\infty & \text{if } N \leq 11, \\ 1 < p < p_{JL}(N-1) & \text{if } N \geq 12. \end{cases}$$

Then,

$$u \equiv 0.$$

where $p_{JL}(N) := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$ is the Joseph-Lundgren exponent.

Existing results

Theorem (Dupaigne-Farina (2022))

Let $\Omega \subset \mathbb{R}^N$ denote a locally Lipschitz-continuous coercive epigraph and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded solution of

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{SPE+}$$

Assume that $f \in C^1([0, +\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$. Then $f(0) = 0$ and $u \equiv 0$.

Idea of the proof

Boundedness of u and monotonicity of u imply that

$$v(x_1, \cdots x_{N-1}) = \lim_{x_N \to +\infty} u(x', x_N),$$

exists and is a classical stable [2] solution to

$$-\Delta v = f(v) \text{ in } \mathbb{R}^{N-1}.$$

Finally, the authors used Liouville-type results established in the whole-space.

2. that is, for every $\phi \in C_c^1(\mathbb{R}^{N-1})$, we have

$$\int_{\mathbb{R}^{N-1}} f'(v)\phi^2 \leq \int_{\mathbb{R}^{N-1}} |\nabla \phi|^2.$$

Our contributions

Theorem (B.-Farina-Sciunzi (2025))

Let Ω be an epigraph bounded from below defined by a function $g \in \mathcal{G}$ and satisfying a uniform exterior cone condition.
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u \geq 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

Assume that $f \in C^1([0, +\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$, then $u \equiv 0$ and $f(0) = 0$.

Theorem (Dupaigne-Farina (2022))

Assume that $u \in C^2(\mathbb{R}^p)$ is bounded below and that u is a stable solution of

$$-\Delta u = f(u) \quad \textit{in } \mathbb{R}^p.$$

where $f : \mathbb{R} :\to \mathbb{R}$ is locally Lipschitz-continuous and nonnegative. If $1 \leq p \leq 10$, then u must be constant.

Theorem (Dupaigne-Farina (2022))

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where $f : \mathbb{R} :\to \mathbb{R}$ is locally Lipschitz-continuous and nonnegative. If $1 \leq p \leq 10$, then u must be constant.

Remarks : This theorem is sharp. Indeed if $p \geq 11$, for $f(u) = u^k$, k sufficiently large, there exists nontrivial positive bounded stable solution to the Lame-Endem equation.

Our contributions

Theorem (B.-Farina-Sciunzi (2025))

Assume $N \geq 12$ and let Ω be an epigraph bounded from below defined by a function $g \in \mathcal{G}$ and satisfying a uniform exterior cone condition.
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u \geq 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

where $f \in C^1([0,+\infty))$ satisfies $f(t) > 0$ for $t > 0$ and $\liminf_{t \to 0^+} \frac{f(t)}{t^s} > 0,$ *for some $s \in \left[0, \frac{N-3}{N-5}\right)$.*
Then $u \equiv 0$ and $f(0) = 0$.


Thank You
For Your Attention
MATH

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

There exists $\delta \in (0, \frac{\tilde{t}}{2})$ such that

$$\forall\, k \geq 1 \quad \exists\, \varepsilon_k \in \left(0, \frac{1}{k}\right),\ \exists\, x^k \in \overline{\Sigma^{g}_{\delta,\tilde{t}-\delta}} \quad : \quad u(x^k) > u_{\tilde{t}+\epsilon_k}(x^k),$$

and so

$$0 \leq g((x^k)') < g((x^k)') + \delta \leq x_N^k \leq \tilde{t} - \delta.$$

thus $x_N^k \to x_\infty$.
We define the sequence

$$v_k(x) = \frac{\tilde{u}(x' + (x^k)', x_N)}{u(x^k)}$$

where $\tilde{u}$ is the extension of u by 0 outside Ω.

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

$$\begin{cases} -\Delta v_k = f_k(v_k) := \frac{f(u(x^k)v_k)}{u(x^k)} & \text{in} \quad \Omega^k, \\ v_k > 0 & \text{in} \quad \Omega^k, \\ v_k = 0 & \text{on} \quad \partial\Omega^k, \\ & \end{cases}$$

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

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Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

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v_k = 0 & \text{on} \quad \partial\Omega^k, \\
v_k(0', x_N^k) = 1, & \\
v_k(0', x_N^k) > v_{k,\tilde{t}+\epsilon_k}(0', x_N^k) & \qquad\qquad \text{(contradiction assumption)}, \\
v_k \leq v_{k,\tilde{t}} & \text{in} \quad \Sigma_{\tilde{t}}^{g_k} \qquad \text{since } u \leq u_{\tilde{t}} \text{ in } \Sigma_{\tilde{t}}^{g}.
\end{cases}
$$

where $\Omega^k := \{x_N > g_k(x')\}$ and $\Sigma_{\tilde{t}}^{g_k} = \{g_k(x') < x_N < \tilde{t}\}$ with $g_k(x') = g(x' + (x^k)')$.

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

We can prove

$$\exists g_\infty \in C^0(\mathbb{R}^{N-1}) \text{ such that } g_k \to g_\infty \text{ in } C^0_{\text{loc}}(\mathbb{R}^{N-1}),$$

$$\exists f_\infty \in Lip([0, +\infty)) \text{ such that } f_k \to f_\infty \text{ in } C^0_{\text{loc}}([0, +\infty)),$$

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

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and let $\mathcal{K} := \overline{B'(0',r) \times (-T,T)}$, $(T > 2\tilde{t})$ we can prove that,

$$\exists v_\infty \in C^0(\mathcal{K}) \text{ such that } v_k \to v_\infty \text{ in } C^0_{\text{loc}}(\mathcal{K}).$$

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$$\exists v_\infty \in C^0(\mathcal{K}) \text{ such that } v_k \to v_\infty \text{ in } C^0_{\text{loc}}(\mathcal{K}).$$

$$\begin{cases} -\Delta v_\infty = f_\infty(v_\infty) & \text{in} \quad \mathcal{C}^{g_\infty}(0',r,T), \\ v_\infty \geq 0 & \text{in} \quad \mathcal{C}^{g_\infty}(0',r,T), \\ v_\infty = 0 & \text{on} \quad \mathcal{C}^{g_\infty}(0',r,T) \cap \{x_N = g_\infty\}, \\ v_\infty(0',x_\infty) = 1, & \\ v_\infty(0',x_\infty) = v_{\infty,\tilde{t}}(0',x_\infty) & \quad \text{since } (0',x_\infty) \in \Sigma_{\tilde{t}}^{g_\infty}. \end{cases}$$

where $\mathcal{C}^{g_\infty}(0',r,T) = B'(0',r) \times (-T,T) \cap \{x_N > g_\infty\}$ (see on board)

Step 2 : $\tilde{t} := \sup \Lambda = +\infty$

By applying the maximum principle to $v_{\infty,\tilde{t}} - v_{\infty}$ in $\Sigma_{\tilde{t}}^{g_\infty}$, we get

$$v_\infty = v_{\infty,\tilde{t}} \text{ in } \Sigma_{\tilde{t}}^{g_\infty}. \tag{3}$$

By applying the maximum principle to v_∞ in $\mathcal{C}^{g_\infty}(0', r, T)$, we get

$$\begin{cases} -\Delta v_\infty \geq f_\infty(v_\infty) - f(0) \geq -L_{f_\infty} v_\infty & \text{in} \quad \mathcal{C}^{g_\infty}(0', r, T), \\ v_\infty \geq 0 & \text{in} \quad \mathcal{C}^{g_\infty}(0', r, T), \\ v_\infty(0', x_\infty) = 1, & \\ 0 = v_\infty(0', g_\infty(0')) = v_\infty(0', 2\tilde{t} - g_\infty(0')) & \text{by (3).} \end{cases}$$