Non linear elliptic problems in unbounded domains

Nicolas Beuvin

PHD's seminar

11 october 2023

UNIVERSITÉ de Picardie Jules Verne

LAMFA

1. Introduction
2. Maximum principle
3. The Half space
    - Monotocity of the solution
    - Classification of stable solutions in lower dimensions
4. The epigraph
    - Coercive epigraph
    - Epigraph

Nonlinear Poisson's equation :

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega \\ u \geqslant 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where

- $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$
- $\Omega \subset \mathbb{R}^N$ a domain bounded or not
- $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$.

Aims

- Monotocity
- Radial symmetry
- One-dimensional symmetry

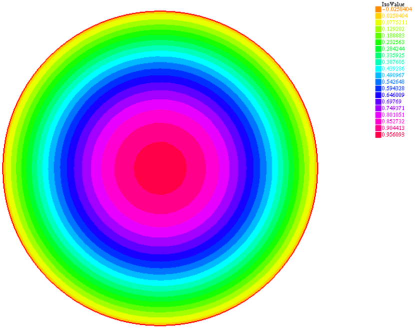
Existing works

Theorem (Gidas,Ni and Nirenberg (79'))

Let $u \in C^2(B(0,1)) \cap C^0(\overline{B(0,1)})$ *which solves* (1), *then* u *is radial ; that is*

$$u(x) = v(r) \ \ (r = |x|)$$

for some strictly decreasing function $v : [0,1] \to [0,+\infty)$.



Title – Level line of the solution of $-\Delta u = u - u^3$ (Freefem ++)

Existing works

Theorem (J.Serrin, H.Zou (98'))

Let $u \in C^2(\mathbb{R}^N)$ *which solves* (1), *moreover we suppose that*

- $\lim_{\|x\| \to +\infty} u(x) = 0$
- $f(0) \geqslant 0$
- $f \in C^{0,1}_{loc}([0 + \infty))$ *and* f *is decreasing on* $[0, \delta]$ *with* $0 < \delta$,

then either $u \equiv 0$ *or* $u > 0$ *and* u *is radially symmetric about some point and stricly radially decreasing.*

1 Introduction

2 Maximum principle

3 The Half space
- Monotocity of the solution
- Classification of stable solutions in lower dimensions

4 The epigraph
- Coercive epigraph
- Epigraph

Theorem (Hopf's lemma)

Let $\Omega \subset \mathbb{R}^N$ be a domain and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta u + cu \geqslant 0 & in \quad \Omega \\ u \geqslant 0 & in \quad \Omega \end{cases}$$

Then

1. *If there exists $x_0 \in \Omega$ such that $u(x_0) = 0$ then*

$$u \equiv 0 \quad in \quad \Omega.$$

2. *Ifnot*

$$u > 0 \quad in \quad \Omega,$$

and if $y_0 \in \partial\Omega$, $u(y_0) = 0$, and Ω satisfies the interior ball condition at y_0 then

$$\frac{\partial u}{\partial \nu}(y_0) < 0.$$

where ν is the exterior unit normal to Ω at y_0.

1 Introduction

2 Maximum principle

3 The Half space
- Monotocity of the solution
- Classification of stable solutions in lower dimensions

4 The epigraph
- Coercive epigraph
- Epigraph

1. Introduction

2. Maximum principle

3. The Half space
   - Monotocity of the solution
   - Classification of stable solutions in lower dimensions

4. The epigraph
   - Coercive epigraph
   - Epigraph

Theorem (A.Farina,2020)

Assume $N \geqslant 2$ *and* $f \in C^{0,1}_{loc}(\mathbb{R}^+)$ *with* $f(0) \geqslant 0$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+ \\ u > 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ = \{x_N = 0\}. \end{cases}$$

Assume that u *is bounded on the slabs* $\mathbb{R}^{N-1} \times [0,t]$ *for every* $t > 0$. *Then*

$$u \text{ is monotone that is } \frac{\partial u}{\partial x_N}(x) > 0 \ \forall x \in \mathbb{R}^N_+.$$

Definition/Notation

Let $0 < \lambda$,we define

- $P_\lambda = \{x \in \mathbb{R}^N, x_N = \lambda\}$
- $\Sigma_\lambda = \{x = (x', x_N) \in \mathbb{R}^N, 0 < x_N < \lambda\}$
- $u_\lambda(x) = u(x_1, \cdots, 2\lambda - x_N)$ the symmetric of u with respect to P_λ

Sketch of the proof

We want to prove that

$$\Gamma := \{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda \ \forall \lambda \leqslant t\} = (0, +\infty).$$

Sketch of the proof

We want to prove that

$$\Gamma := \{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda \; \forall \, \lambda \leqslant t\} = (0, +\infty).$$

Indeed, for $t > 0$, if we define on Σ_t, $w_t = u_t - u$ then w_t satisfy

$$\begin{cases} -\Delta w_t + L_{f,[0,\|u\|_{\Sigma_{2t}}]} w_t \geqslant 0 & \text{in} \quad \Sigma_t \\ w_t \geqslant 0 & \text{in} \quad \Sigma_t \\ w_t = 0 & \text{on} \quad \{x = (x', x_N) \in \mathbb{R}^N, x_N = t\}. \end{cases}$$

Sketch of the proof

We want to prove that

$$\Gamma := \{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda \ \forall \lambda \leqslant t\} = (0, +\infty).$$

Indeed, for $t > 0$, if we define on Σ_t, $w_t = u_t - u$ then w_t satisfy

$$\begin{cases} -\Delta w_t + L_{f,[0,\|u\|_{\Sigma_{2t}}]} w_t \geqslant 0 & \text{in} \quad \Sigma_t \\ w_t \geqslant 0 & \text{in} \quad \Sigma_t \\ w_t = 0 & \text{on} \quad \{x = (x', x_N) \in \mathbb{R}^N, x_N = t\}. \end{cases}$$

By Hopf's Lemma :

$$\forall x' \in \mathbb{R}^{N-1} \qquad -2\frac{\partial u}{\partial x_N}(x', t) = \frac{\partial w_t}{\partial x_N}(x', t) < 0.$$

$\Gamma \neq \emptyset$

Theorem (Comparison principle in unbounded slabs of smalls width)

Let $N \geqslant 2$, $M > 0$,$f \in C^{0,1}_{loc}(\mathbb{R}^+)$ and $a < b$. Let $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$ satisfying

$$\begin{cases} -\Delta u - f(u) \leqslant -\Delta v - f(v) & \text{in } \mathbb{R}^{N-1} \times (a, b) \\ |u|, |v| < M & \text{in } \mathbb{R}^{N-1} \times (a, b) \\ u \leqslant v & \text{on } \partial(\mathbb{R}^{N-1} \times (a, b)). \end{cases}$$

Then there exist $\theta = \theta(f, M) > 0$ such that and any

$$0 < b - a < \theta \Rightarrow u \leqslant v \text{ in } \mathbb{R}^{N-1} \times (a, b).$$

$\Gamma \neq \emptyset$

Theorem (Comparison principle in unbounded slabs of smalls width)

Let $N \geqslant 2$, $M > 0$, $f \in C^{0,1}_{loc}(\mathbb{R}^+)$ *and* $a < b$. *Let* $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$ *satisfying*

$$\begin{cases} -\Delta u - f(u) \leqslant -\Delta v - f(v) & \text{in } \mathbb{R}^{N-1} \times (a, b) \\ |u|, |v| < M & \text{in } \mathbb{R}^{N-1} \times (a, b) \\ u \leqslant v & \text{on } \partial(\mathbb{R}^{N-1} \times (a, b)). \end{cases}$$

Then there exist $\theta = \theta(f, M) > 0$ *such that and any*

$$0 < b - a < \theta \Rightarrow u \leqslant v \text{ in } \mathbb{R}^{N-1} \times (a, b).$$

We define

$$\tilde{t} = \sup\{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda \; \forall \lambda \leqslant t\}.$$

if $\tilde{t} < +\infty$

Proposition

For every $\delta \in (0, \frac{\tilde{t}}{2})$*, there exists* $\varepsilon(\delta) > 0$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \; u \leqslant u_{\tilde{t}+\varepsilon} \text{ in } \mathbb{R}^{N-1} \times [\delta, \tilde{t} - \delta].$$

if $\tilde{t} < +\infty$

Proposition

For every $\delta \in (0, \frac{\tilde{t}}{2})$*, there exists* $\varepsilon(\delta) > 0$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \; u \leqslant u_{\tilde{t}+\varepsilon} \; \textit{in} \; \mathbb{R}^{N-1} \times [\delta, \tilde{t} - \delta].$$

so we have

$$\begin{cases} u \leqslant u_{\tilde{t}+\varepsilon} & \text{on} \quad \mathbb{R}^{N-1} \times [\delta, \tilde{t} - \delta] \\ u \leqslant u_{\tilde{t}+\varepsilon} & \text{on} \quad \mathbb{R}^{N-1} \times [0, \delta] \\ u \leqslant u_{\tilde{t}+\varepsilon} & \text{on} \quad \mathbb{R}^{N-1} \times [\tilde{t} - \delta, \tilde{t} + \varepsilon]. \end{cases}$$

1 Introduction

2 Maximum principle

3 The Half space
- Monotocity of the solution
- Classification of stable solutions in lower dimensions

4 The epigraph
- Coercive epigraph
- Epigraph

Stable solution

Definition (Stable solution)

Let $f \in C^1(\mathbb{R})$ and let Ω denote an open set of $\mathbb{R}^N$, $N \geqslant 1$. A solution $u \in C^2(\Omega)$ of

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{2}$$

is stable if

$$\int_\Omega f'(u)\phi^2 dx \leqslant \int_\Omega \|\nabla \phi\|^2 dx, \quad \forall\ \phi \in C_c^1(\Omega).$$

Stable solution

Definition (Stable solution)

Let $f \in C^1(\mathbb{R})$ and let Ω denote an open set of $\mathbb{R}^N$, $N \geqslant 1$. A solution $u \in C^2(\Omega)$ of

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{2}$$

is stable if

$$\int_\Omega f'(u)\phi^2 dx \leqslant \int_\Omega \|\nabla \phi\|^2 dx, \quad \forall\, \phi \in C_c^1(\Omega).$$

Proposition

Let $u \in C^2(\Omega)$ solution of (2) such that,

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{in } \Omega$$

Then u is stable.

Stable solution

Theorem (L.Dupaigne and A.Farina (2020))

Let $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a bounded solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N_+ \\ u > 0 & \text{in} \quad \mathbb{R}^N_+ \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{3}$$

Assume that $f \in C^1([0, +\infty))$ *and non negative.*
If $2 \leqslant N \leqslant 11$ *then* u *must be one-dimensional and monotone (ie* $u = u(x_N)$ *and* $\partial u / \partial x_N > 0$ *in* $\mathbb{R}^N_+$*).*

Stable solution

Theorem (L.Dupaigne and A.Farina (2020))

Assume that $u \in C^2(\mathbb{R}^N)$ is bounded below and that u is a stable solution of

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $f \in C^1([0, +\infty))$ is locally lipschitz and non negative.
If $N \leqslant 10$ then u must be constant.

Theorem (H.Berestycki, L.A.Caffarelli and L.Nirenberg (93'))

Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution of (3). If

$$f(\sup_{\mathbb{R}^N_+}(u)) \leqslant 0$$

then

$$u \text{ is a function of } x_N.$$

1 Introduction

2 Maximum principle

3 The Half space
- Monotocity of the solution
- Classification of stable solutions in lower dimensions

4 The epigraph
- Coercive epigraph
- Epigraph

1. Introduction
2. Maximum principle
3. The Half space
   - Monotocity of the solution
   - Classification of stable solutions in lower dimensions
4. The epigraph
   - Coercive epigraph
   - Epigraph

Coercive epigraph

Definition (Coercive epigraph)

A domain Ω is a coercive epigraph if there exists $g \in C^0(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\Omega = \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\}$ and $\lim_{\|x'\| \to +\infty} g(x') = +\infty$.

Coercive epigraph

Definition (Coercive epigraph)

A domain Ω is a coercive epigraph if there exists $g \in C^0(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\Omega = \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\}$ and $\lim_{\|x'\| \to +\infty} g(x') = +\infty$.

Theorem (M.J.Esteban and P.L.Lions (82'))

Let Ω denote a coercive epigraph. If $u > 0$ solves (1) then

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{in } \Omega.$$

1. Introduction

2. Maximum principle

3. The Half space
   - Monotocity of the solution
   - Classification of stable solutions in lower dimensions

4. The epigraph
   - Coercive epigraph
   - Epigraph

Epigraph

Theorem

Let $N \geqslant 2$, $g \in UC(\mathbb{R}^{N-1})$ *such that* g *is bounded below . We consider* Ω *the epigraph of* g. *Moreover, let* $f \in C^{0,1}([0,+\infty))$ *such that* $f(0) > 0$. *Let* $u \in UC(\overline{\Omega}) \cap C^2(\Omega) \cap H^1_{loc}(\overline{\Omega})$ *a function satisfying*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \\ u \geqslant 0 & \text{in} \quad \Omega \end{cases}$$

then u *is monotone, that is*

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \forall x \in \Omega.$$

Definition/Notation

Let $0 < a < b$ and $g \in C^0(\mathbb{R}^{N-1})$, we introduce

- $P_b = \{x \in \mathbb{R}^N, x_N = b\}$
- $\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') < x_N < b\}$
- $\Sigma_{a,b}^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') + a < x_N < b\}$
- $u_b(x) = u(x_1, \cdots, 2b - x_N)$ the symmetric of u with respect to P_b.

Definition/Notation

Let $0 < a < b$ and $g \in C^0(\mathbb{R}^{N-1})$, we introduce

- $P_b = \{x \in \mathbb{R}^N, x_N = b\}$
- $\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') < x_N < b\}$
- $\Sigma_{a,b}^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') + a < x_N < b\}$
- $u_b(x) = u(x_1, \cdots, 2b - x_N)$ the symmetric of u with respect to P_b.

We define

$$\Gamma := \{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda^g \;\forall \lambda \leqslant t\}.$$

Comparison principle

Theorem

Let $N \geqslant 2$ and Ω an open set included in a strip $\{x = (x', x_N) \in \mathbb{R}^N, a < x_N < b\}$ with $a < b$. Let $f \in C^{0,1}(\mathbb{R})$, $\gamma > 0$ and $u, v \in H^1_{loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} -\Delta u - f(u) \leqslant -\Delta v - f(v) & \text{in} \quad D'(\Omega) \\ |u|, |v| \leqslant \gamma e^{\|x\|} & \text{in} \quad \Omega \\ u \leqslant v & \text{on} \quad \partial\Omega. \end{cases}$$

then there exist $\varepsilon = \varepsilon(L_f) > 0$ such that

$$0 < L(\Omega) < \varepsilon \implies u \leqslant v \text{ in } \Omega$$

where $L(\Omega) = \sup_{\mathbb{R}^{N-1}} (\mathcal{L}^1((\{x'\} \times \mathbb{R}) \cap \Omega))$.

Comparison principle

Theorem

Let $N \geqslant 2$ and Ω an open set included in a strip $\{x = (x', x_N) \in \mathbb{R}^N, a < x_N < b\}$ with $a < b$. Let $f \in C^{0,1}(\mathbb{R})$, $\gamma > 0$ and $u, v \in H^1_{loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} -\Delta u - f(u) \leqslant -\Delta v - f(v) & \text{in} \quad D'(\Omega) \\ |u|, |v| \leqslant \gamma e^{\|x\|} & \text{in} \quad \Omega \\ u \leqslant v & \text{on} \quad \partial\Omega. \end{cases}$$

then there exist $\varepsilon = \varepsilon(L_f) > 0$ such that

$$0 < L(\Omega) < \varepsilon \implies u \leqslant v \text{ in } \Omega$$

where $L(\Omega) = \sup_{\mathbb{R}^{N-1}} (\mathcal{L}^1((\{x'\} \times \mathbb{R}) \cap \Omega))$.

We define

$$\tilde{t} = \sup\{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda^g \ \forall \lambda \leqslant t\}.$$

If $\tilde{t} < +\infty$

Proposition

For every $\delta \in (0, \frac{\tilde{t}}{2})$*, there exists* $\varepsilon(\delta) > 0$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \ u \leqslant u_{\tilde{t}+\varepsilon} \text{ in } \Sigma^g_{\delta,\tilde{t}-\delta}.$$

so we have

$$\begin{cases} u \leqslant u_{\tilde{t}+\varepsilon} & \text{on} \quad \Sigma^g_{\delta,\tilde{t}-\delta} \\ u \leqslant u_{\tilde{t}+\varepsilon} & \text{on} \quad \Sigma^g_{\tilde{t}+\varepsilon} \backslash \Sigma^g_{\delta,\tilde{t}-\delta}. \end{cases}$$


Thank You
For Your Attention
MATH

1 Introduction

2 Maximum principle

3 The Half space
- Monotocity of the solution
- Classification of stable solutions in lower dimensions

4 The epigraph
- Coercive epigraph
- Epigraph

1. Berestycki, H., Caffarelli, L. A. & Nirenberg, L. Monotonicity for elliptic equations in unbounded Lipschitz domains. en. *Communications on Pure and Applied Mathematics* **50,** 1089-1111. ISSN : 1097-0312. (2023) (1997).

2. Berestycki, H., Caffarelli, L. A. & Nirenberg, L. English. in *Boundary value problems for partial differential equations and applications. Dedicated to Enrico Magenes on the occasion of his 70th birthday* 27-42 (Masson, Paris, 1993). ISBN : 978-2-225-84334-1.

3. Dupaigne, L. *Stable Solutions of Elliptic Partial Differential Equations.* ISBN : 978-0-429-15034-0 (Chapman et Hall/CRC, New York, mai 2011).

4. Esteban, M. J. & Lions, P. L. Existence and non-existence results for semilinear elliptic problems in unbounded domains. en. *Proceedings of the Royal Society of Edinburgh : Section A Mathematics* **93,** 1-14. ISSN : 0308-2105, 1473-7124. https://www.cambridge.org/core/product/identifier/S0308210500031607/type/journal_article (2023) (1982).

5. Evans, L. C. *Partial differential equations.* en (1988).

6. Farina, A. Some results about semilinear elliptic problems on half-spaces. en. *Mathematics in Engineering* **2,** 709-721. ISSN : 2640-3501. http://www.aimspress.com/article/10.3934/mine.2020033 (2023) (2020).

7. Gilbarg, D. & Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order.* ISBN : 978-3-540-41160-4 (Springer Science & Business Media, jan. 2001).

8. SERRIN, J. & ZOU, H. Symmetry of Ground States of Quasilinear Elliptic Equations. en. *Archive for Rational Mechanics and Analysis* **148,** 265-290. ISSN : 1432-0673. https://doi.org/10.1007/s002050050162 (2023) (sept. 1999).