Serrin's overdetermined problem in epigraphs

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LAMFA

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where

- $c \in \mathbb{R}\backslash\{0\}$
- $u \in C^2(\overline{\Omega})$ is a classical solution.
- $\Omega \subset \mathbb{R}^N$ is an epigraph, i.e

$$\{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$

  where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a differentiable function.
- $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function on $\mathbb{R}$.

Historic

A lot of applications in Physics : fluid mechanics,...

Example : The Soap bubble problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega, \\ u \geqslant 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \eta} = c & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

where

- u represent a fluid inside a soap bubble.
- c is a constant related to the viscosity and density of u.
- Ω is a soap bubble.

Historic

J. Serrin proved in 1971 the following result :

Theorem (Soap bubble Theorem)

Let Ω be a bounded domain whose boundary is of class C^2. If there exists a function $u \in C^2(\overline{\Omega})$ satisfying (1) then Ω must be a ball and u is radially symmetric about its center.

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Theorem (Soap bubble Theorem)

Let Ω be a bounded domain whose boundary is of class C^2. If there exists a function $u \in C^2(\overline{\Omega})$ satisfying (1) then Ω must be a ball and u is radially symmetric about its center.

Question : What is the situation when Ω is an unbounded domain ?

Historic

In 1997, H. Berestycki, L. Caffarelli and L. Nirenberg conjectured that

Conjecture

If Ω is a smooth domain with Ω^c connected and that there is a bounded positive solution of (1) *for some Lipschitz function f then Ω is either a half space, or a cylinder $\Omega = B_k \times \mathbb{R}^{n-k}$, where B_k is k-dimensional Euclidean ball, or the complement of a ball or a cylinder.*

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In our case, the conjecture becomes

Conjecture

If Ω is a smooth enough epigraph and that there is a bounded positive solution of (1) for some Lipschitz function f then Ω is a half space.

Theorem (case $f(0) \geq 0$)

Let Ω be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition on $\partial\Omega$. Assume $f \in C^{0,1}_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u > 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

Suppose that $\nabla u \in L^\infty(\Omega)$, then u is strictly increasing in the x_N-direction, i.e.

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \forall x \in \Omega.$$

Definition

We say that Ω satisfy a uniform exterior cone condition on $\partial\Omega$ if for any $x_0 \in \partial\Omega$ there exists a finite right circular cone V_{x_0} with vertex x_0 such that

$$\overline{\Omega} \cap V_{x_0} = \{x_0\},$$

and the cones V_{x_0} are all congruent to some fixed cone V.

Definition

Examples :

- If Ω is a Lipschitz epigraph (i.e g is a Lipschitz continuous function) then Ω satisfy a uniform exterior cone condition on $\partial\Omega$.

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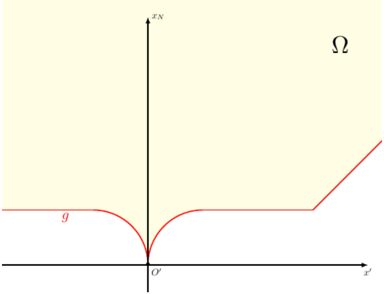
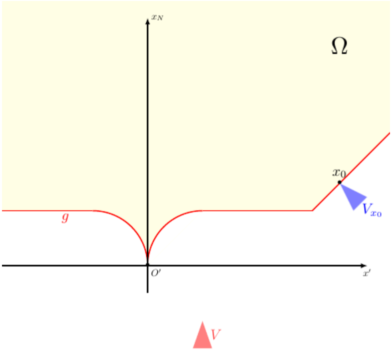


$$g(x) = \begin{cases} 2 & x \in]-\infty, -2], \\ \sqrt{4-(x+2)^2} & x \in]-2, 0], \\ \sqrt{4-(x-2)^2} & x \in]0, 2], \\ 2 & x \in]2, 6], \\ x-4 & x \in]6, +\infty[. \end{cases}$$

Definition



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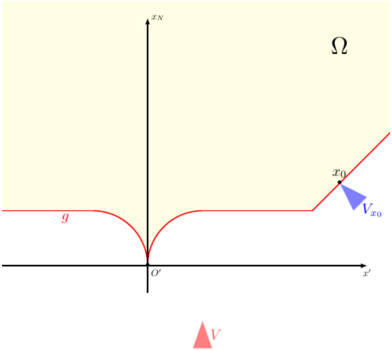

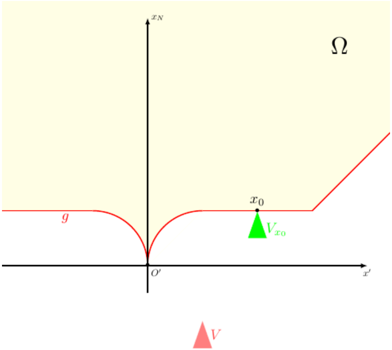

Remark

We don't necessarily need that the epigraph be uniformly continuous. Indeed, if g satisfy the following property then the Theorem "case $f(0) \geq 0$" holds true.

Proposition

There exists an injection $\phi : g(\mathbb{R}^{N-1}) \to \mathbb{R}$ continuous such that $\phi \circ g$ is uniformly continuous on $\mathbb{R}^{N-1}$.

Example :

- $$\exp : \mathbb{R}^{N-1} \to \mathbb{R}$$
  $$x' = (x_1, \cdots, x_{N-1}) \to e^{x_1}.$$

Theorem (case $f(0) < 0$)

Let $N \geq 2$ and Ω be a continuous epigraph bounded from below such that $g \in \mathfrak{C}^{1,\alpha}(\mathbb{R}^{N-1})$. Let $f \in C^1(\mathbb{R})$ such that $f(0) < 0$ and $u \in C^2(\overline{\Omega})$ be a bounded solution to

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u \geq 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \\ \frac{\partial u}{\partial \eta} = c & on \quad \partial\Omega. \end{cases}$$

Then u is strictly increasing in the x_N-direction, i.e.

$$\frac{\partial u}{\partial x_N} > 0 \quad in \quad \Omega.$$

Definition/Remark

Definition

Let $X \subset \mathbb{R}^N$ an open set and $\alpha \in (0,1]$. We denote by $\mathfrak{C}^{1,\alpha}(X)$ the set of functions $h : X \to \mathbb{R}$ in such a way that $h \in C^1(X)$ and $\nabla h \in C^{0,\alpha}(X)$, that is

$$\|\nabla h\|_{C^{0,\alpha}(X)} = \sup_{x \in \overline{X}} |\nabla h(x)| + \sup_{x,y \in \overline{X}, x \neq y} \frac{|\nabla h(x) - \nabla h(y)|}{|x-y|^\alpha} < +\infty.$$

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Remark :

Theorem

Let $\Omega \subset \mathbb{R}^N$ be an epigraph with $g \in C^1(\mathbb{R}^{N-1})$. Let $f \in C^0(\mathbb{R})$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a bounded solution to (1). Then

$$\nabla u \text{ is bounded in } \Omega,$$

If $f(0) \geq 0$ then the Theorem "case $f(0) < 0$" is true since g is Lipschitz continuous, bounded from below and ∇u is bounded.

Definition

Let $0 < a < b$, we define

- $\Sigma_a^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') < x_N < a\}$,
- $\Sigma_{a,b}^g = \{x = (x', x_N) \in \mathbb{R}^N, g(x') + a < x_N < b\}$,

$$\forall x \in \Sigma_a^g \quad u_a(x) = u(x_1, \cdots, 2a - x_N).$$

Sketch of the proof

We want to prove that

$$\Gamma := \{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda^g \; \forall \lambda \leqslant t\} = (0, +\infty).$$

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Indeed, for $t > 0$, if we define on Σ_t^g, $w_t = u_t - u$ then w_t satisfy

$$\begin{cases} -\Delta w_t + L_{f,[0,\|u\|_{\Sigma_{2t}^g}]} w_t \geqslant 0 & \text{in} \quad \Sigma_t^g, \\ w_t \geqslant 0 & \text{in} \quad \Sigma_t^g, \\ w_t = 0 & \text{on} \quad \{x = (x', x_N) \in \Omega, x_N = t\}. \end{cases}$$

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By Hopf's Lemma :

$$\forall x' \in \mathbb{R}^{N-1} \quad -2\frac{\partial u}{\partial x_N}(x', t) = \frac{\partial w_t}{\partial x_N}(x', t) < 0.$$

$\Gamma \neq \emptyset$

Theorem (Comparison principle in unbounded slabs of smalls width)

Let $N \geqslant 2$, $\Omega = \Sigma_t^g$ an open set included in a strip $\mathbb{R}^{N-1} \times [0, b]$ ie $t < b$, $M > 0$, $f \in C^{0,1}_{loc}(\mathbb{R}^+)$. Let $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u - f(u) \leqslant -\Delta v - f(v) & in \quad \Omega, \\ |u|, |v| < M & in \quad \Omega, \\ u \leqslant v & on \quad \partial\Omega. \end{cases}$$

Then there exist $\theta = \theta(f, M) > 0$ such that

$$0 < t < \theta \Rightarrow u \leqslant v \ in \ \Omega.$$

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Then there exist $\theta = \theta(f, M) > 0$ *such that*

$$0 < t < \theta \Rightarrow u \leqslant v \text{ in } \Omega.$$

We define

$$\tilde{t} := \sup\{t > 0, u \leqslant u_\lambda \text{ in } \Sigma_\lambda^g \ \forall \lambda \leqslant t\}.$$

if $\tilde{t} < +\infty$

Proposition

For every $\delta \in (0, \frac{\tilde{t}}{2})$*, there exists* $\varepsilon(\delta) \in (0, \delta)$ *such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \quad u \leqslant u_{\tilde{t}+\varepsilon}, \text{ in } \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}.$$

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so we have

$$\begin{cases} u \leqslant u_{\tilde{t}+\varepsilon} & \text{in} \quad \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}, \\ u \leqslant u_{\tilde{t}+\varepsilon} & \text{in} \quad \Sigma^g_{\tilde{t}+\varepsilon} \backslash \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}. \end{cases}$$

Theorem (A.Farina, E.Valdinoci, 2009)

Let $N = 2, 3$ and f be locally Lipschitz.
Let Ω be an open epigraph of $\mathbb{R}^N$ with C^3 and uniformly Lipschitz boundary.
Suppose that $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (1) and that there exists $\delta_3 > \delta_2 > \delta_1 > 0$ in such a way that

- $f(t) > \delta_1 t$ *for any* $t \in (0, \delta_1)$,
- f *is nonincreasing on* (δ_2, δ_3),
- $f > 0$ *on* $(0, \delta_3)$,
- $f \leq 0$ *on* $[\delta_3, +\infty)$.

Then, we have that $\Omega = \mathbb{R}^N_+$ up to isometry and that there exists $u_0 : (0, +\infty) \to (0, +\infty)$ in such a way that

$$u(x_1, \cdots, x_N) = u_0(x_N) \quad \text{for any } (x_1, \cdots, x_N) \in \Omega.$$

Theorem (Case $N = 2$)

Let $N = 2$ and $\Omega \subset \mathbb{R}^N$ an epigraph bounded from below such that $g \in C^3(\mathbb{R}^{N-1}) \cap \mathfrak{C}^{1,\alpha}(\mathbb{R}^{N-1})$. Let $f \in C^1([0, +\infty))$ and u a bounded solution to (1).
Then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ strictly increasing such that

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Remark : If $f(0) \geq 0$ then we can just suppose that Ω is an uniformly continuous epigraph bounded from below, of class C^3.

We define

$$F(t) = \int_0^t f(s)ds \quad \text{and} \quad c_u := \sup_{t \in [0, \sup_\Omega u]} F(t).$$

Theorem (Case $N = 3$)

Let $N = 3$ and $\Omega \subset \mathbb{R}^N$ an epigraph bounded from below such that $g \in C^3(\mathbb{R}^{N-1}) \cap \mathfrak{C}^{1,\alpha}(\mathbb{R}^{N-1})$. Let $f \in C^1([0, +\infty))$ and u a bounded solution to (1).
Suppose that

$$c_u = F(\sup_\Omega u). \tag{3}$$

Then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \quad \forall x \in \Omega.$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz continuous epigraph bounded from below. Let $f \in C^1(\mathbb{R})$ and $u \in C^2(\overline{\Omega})$ be a bounded solution to (1)*.*
Assume that one of the following assumptions hold true :

(H1) $f(t) \geq 0$*, for any* $t \geq 0$*,*

(H2) there exists $\zeta > 0$*, such that* $f(t) \geq 0$ *on* $[0, \zeta]$ *and* $f(t) \leq 0$ *on* $[\zeta, +\infty)$*,*

(H3) there exists $0 < \zeta_1 < \zeta_2$ *such that* $f(t) \geq 0$ *in* $[0, \zeta_1]$*,* $f(t) \leq 0$ *in* $[\zeta_1, \zeta_2]$ *and* $f(t) > 0$ *in* $(\zeta_2, +\infty)$*.*
Then,

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Then,

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Theorem

Let $\Omega \subset \mathbb{R}^3$ be an epigraph bounded from below with $g \in C^{2,\alpha}_{loc}(\mathbb{R}^2)$ such that $\|\nabla g\|_{C^{1,\alpha}(\mathbb{R}^2)} < +\infty$,
$f \in C^1(\mathbb{R})$ such that $f(0) \geq 0$ and $u \in C^2(\overline{\Omega})$ be a bounded solution to (1)*.*
Then,

$$c_u = F(\sup_\Omega u).$$

A Poincaré-geometric's type formule

Theorem

Let Ω be an open subset of $\mathbb{R}^N$ with C^3 boundary.
Suppose that $u \in C^2(\overline{\Omega})$ satisfies (1) with f locally Lipschitz, and

$$\frac{\partial u}{\partial x_N}(x) > 0, \qquad \text{for any } x \in \Omega.$$

Then, for any $R > 0$

$$\int_{\Omega \cap B_{\sqrt{R}}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2) \leq \frac{4}{\ln(R)^2} \int_{B_R \setminus B_{\sqrt{R}} \cap \Omega} \frac{|\nabla u(x)|^2}{|x|^2} dx, \tag{4}$$

and

$$\int_{B_R \setminus B_{\sqrt{R}} \cap \Omega} \frac{|\nabla u(x)|^2}{|x|^2} dx \leq \int_{\sqrt{R}}^{R} \int_{B_t \setminus B_{\sqrt{R}} \cap \Omega} \frac{2|\nabla u(x)|^2}{t^3} dx dt + \int_{B_R \setminus B_{\sqrt{R}} \cap \Omega} \frac{|\nabla u|^2}{R^2}. \tag{5}$$

Case $N = 3$.

Lemma

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz continuous epigraph with a boundary of class C^1. Let $f \in C^1(\mathbb{R})$ and $u \in C^2(\overline{\Omega})$ be a bounded solution to (1).
Assume that

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \text{for any } x \in \Omega.$$

and

$$c_u = F(\|u\|_{L^\infty(\Omega)}),$$

Then, there exists $C > 0$, in such a way that

$$\int_{B_R \cap \Omega} |\nabla u(x)|^2 dx \leq CR^2 \quad \text{for any} \quad R > 0.$$


Thank You
For Your Attention
MATH

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

Suppose that there exists $\delta \in (0, \frac{\tilde{t}}{2})$ in such a way that

$$\forall k > 0 \quad \exists \varepsilon_k \in (0, \frac{1}{k}) \quad \exists x^k \in \overline{\Sigma^g_{\delta, \tilde{t}-\delta}} \quad \text{such that } u(x^k) > u_{\tilde{t}+\varepsilon_k}(x^k).$$

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$x^k_N \in [\delta, \tilde{t} - \delta]$, thus $x^k_N \to x_\infty \in [\delta, \tilde{t} - \delta]$.

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Suppose that there exists $\delta \in (0, \frac{\tilde{t}}{2})$ in such a way that

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$x_N^k \in [\delta, \tilde{t} - \delta]$, thus $x_N^k \to x_\infty \in [\delta, \tilde{t} - \delta]$. We fix

$$u_k(x) = u(x' + (x^k)', x_N)$$

where $x = (x', x_N) \in \Omega^k := \{(x', x_N) \in \mathbb{R}^N, x_N > g_k(x')\}$ and

$$g_k(x') = g(x' + (x^k)').$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

We have

$$\begin{cases} -\Delta u_k = f(u_k) & \text{in} \quad \Omega^k \\ u_k > 0 & \text{in} \quad \Omega^k \\ u_k = 0 & \text{on} \quad \partial\Omega^k \\ u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k) & \\ u_k(x) \le u_{k,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_k} \end{cases}$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

We have

$$\begin{cases} -\Delta u_k = f(u_k) & \text{in} \quad \Omega^k \\ u_k > 0 & \text{in} \quad \Omega^k \\ u_k = 0 & \text{on} \quad \partial\Omega^k \\ u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k) & \\ u_k(x) \leq u_{k,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_k} \end{cases}$$

We can show that there exists $g_\infty \in C^0(\mathbb{R}^{N-1})$ such that

$$g_k \to g_\infty \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^{N-1}).$$

We denote by Ω^∞ its epigraph.

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

We have

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And there exists $u_\infty \in C^2(\Omega^\infty)$ such that

$$u_k \to u_\infty \quad \text{in } C^0_{\text{loc}}(\Omega^\infty).$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

Moreover u_∞ solves

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ u_\infty \geqslant 0 & \text{in} \quad \Omega^\infty, \\ u_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \\ u_\infty(0', x_\infty) > u_{\infty,\tilde{t}}(0', x_\infty), & \\ u_\infty(x) \leq u_{\infty,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}. \end{cases}$$

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We have

$$-\Delta u_\infty + L_f u_\infty \geq 0 \quad \text{in } \Omega$$

thus by the maximum principle

$$\text{either } u_\infty \equiv 0 \quad \text{ or either } u_\infty > 0.$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

If we fix $w = u_{\infty,\tilde{t}} - u_\infty$ then we have

$$\begin{cases} -\Delta w + L_f w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w(0', x_\infty) = 0 & \end{cases}$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

If we fix $w = u_{\infty,\tilde{t}} - u_{\infty}$ then we have

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Therefore, by the maximum principle $w \equiv 0$ in connected componant of $\Sigma_{\tilde{t}}^{g_\infty}$ which contains $(0', x_\infty)$.

Theorem (Hopf's lemma)

Let $\Omega \subset \mathbb{R}^N$ be a domain and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta u + cu \geqslant 0 & in \quad \Omega \\ u \geqslant 0 & in \quad \Omega \end{cases}$$

Then

1. *If there exists $x_0 \in \Omega$ such that $u(x_0) = 0$ then*

$$u \equiv 0 \quad in \quad \Omega.$$

2. *Ifnot*

$$u > 0 \quad in \quad \Omega,$$

and if $y_0 \in \partial\Omega$, $u(y_0) = 0$, and Ω satisfies the interior ball condition at y_0 then

$$\frac{\partial u}{\partial \nu}(y_0) < 0.$$

where ν is the exterior unit normal to Ω at y_0.