Symmetry and classification results for solutions of nonlinear Poisson's equation

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PHD's seminar

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- $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is an epigraph, i.e

$$\{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\},$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a differentiable function.

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  where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a differentiable function.
- $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function on $\mathbb{R}$ or Lipschitz-continuous function on $\mathbb{R}$.

Symmetry results general

Theorem (B., Farina, 2025)

Let $\Omega = \mathbb{R}^N_+$ or $\Omega \subset \mathbb{R}^N$ be a domain of class C^3. Let $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^2(\overline{\Omega})$ be a solution to (1) such that

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad \text{as } R \longrightarrow \infty. \tag{2}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{3}$$

then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : \mathbb{R} \to (0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \quad \forall x \in \mathbb{R}^N_+.$$

Monotonicity results

Theorem (B., Farina, Sciunzi, 2025)

Let Ω be a globally Lipschitz continuous epigraph bounded from below, $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let u be a classical solution of

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u > 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{4}$$

Assume that u have at most exponential growth on finite strip then u is strictly increasing in the x_N-direction, i.e.

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \forall x \in \Omega.$$

Monotonicity results

Remark If f is not Lipschitz continuous then Theorem 2 is false. Indeed

$$u(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x_N \leq 1, \\ (1-(x_N-2)^4)^4 & \text{if} \quad 1 < x_N \leq 3, \\ (1-(x_N-4)^4)^4 & \text{if} \quad 3 < x_N \leq 4, \\ 1 & \text{if} \quad x_N > 4, \end{cases}$$

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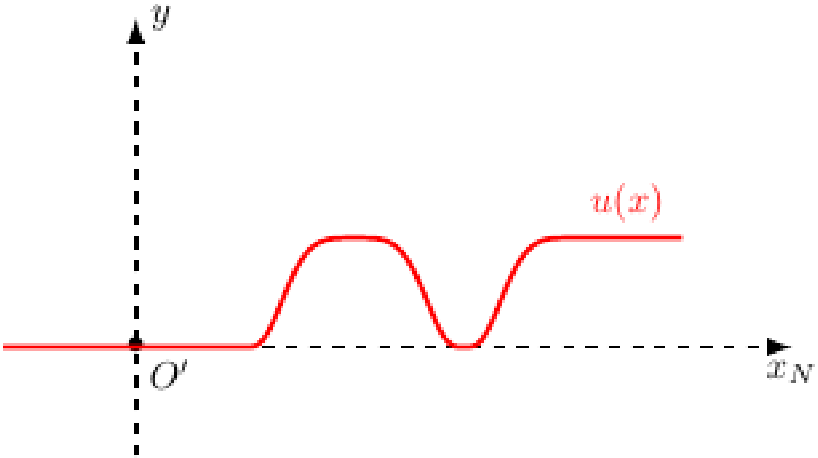
is a classical solution of (4) with

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ -192(t(1-t^{\frac{1}{4}}))^{\frac{1}{2}}(1-\frac{5}{4}t^{\frac{1}{4}}) & \text{if} \quad 0 \le t \le 1, \\ 0 & \text{if} \quad t > 1. \end{cases}$$

Monotonicity results



Monotonicity results

Theorem (B., Farina, 2025)

Let Ω *be a smooth enough epigraph bounded from below,* $f \in Lip_{loc}([0, +\infty))$ *with* $f(0) < 0$ *and let* u *be a classical solution of* (1) *with* $c \neq 0$ *Assume that*

$$\nabla u \text{ is bounded on finite strips.}$$

Then u *is strictly increasing in the* x_N-*direction, i.e.*

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \forall x \in \Omega.$$

Monotonicity results

Theorem (Farina, Sciunzi, 2017)

Let $f \in Lip_{loc}([0,+\infty))$ *and* $u \in C^2(\overline{\mathbb{R}^2_+})$ *be a classical solution of*

$$\begin{cases} -\Delta u = f(u) & in \quad \mathbb{R}^2_+, \\ u > 0 & in \quad \mathbb{R}^2_+, \\ u = 0 & on \quad \partial\mathbb{R}^2_+. \end{cases}$$

Then u *is strictly increasing in the* x_2-*direction, i.e.*

$$\frac{\partial u}{\partial x_2}(x) > 0 \quad \forall x \in \mathbb{R}^2_+.$$

Theorem (B., Farina, 2025)

Let $\Omega \subset \mathbb{R}^2$ *be a smooth enough epigraph bounded from below and let* $u \in C^2(\overline{\Omega})$ *be a classical solution of* (1). *Assume that* $f \in Lip_{loc}([0, +\infty))$, $f(0) \geq 0$ *and*

$$\nabla u \in L^\infty(\Omega);$$

Then, $\Omega = \mathbb{R}^2_+$ *up to a vertical translation and there exists* $u_0 : [0, +\infty) \to (0, +\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

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Remark : if $\nabla u \notin L^\infty(\Omega)$ then Theorem (5) is false (see $u(x_1, x_2) = x_2 e^{x_1}$ in $\mathbb{R}^2_+$)

Theorem (B., Farina, 2025)

Let $\Omega \subset \mathbb{R}^2$ *be a smooth enough epigraph bounded from below and let* $u \in C^2(\overline{\Omega})$ *be a classical solution of* (1). *Assume that :*

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(0) \geq 0, \quad f(t) \leq 0 \quad in \quad (0,+\infty), \\ u(x) = o(|x| \ln^{\frac{1}{2}} |x|), \qquad as \quad |x| \longrightarrow \infty. \end{cases}$$

Then, $\Omega = \mathbb{R}^2_+$ *up to a vertical translation and there exists* $u_0 : [0,+\infty) \to (0,+\infty)$ *strictly increasing such that*

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$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Remark : All previous Theorem hold true even if $c = 0$.

Let $f \in C^1(\mathbb{R})$, we say that a solution u of

$$-\Delta u = f(u) \quad \text{in } \Omega,$$

is stable if, for any $\phi \in C_c^1(\Omega)$, there holds

$$\int_\Omega f'(u)\phi^2 \leq \int_\Omega |\nabla \phi|^2.$$

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Theorem (Dupaigne, Farina, 2022)

Assume that $u \in C^2(\mathbb{R}^N)$ is bounded below and that u is a stable solution of

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where $f : \mathbb{R} :\to \mathbb{R}$ is locally Lipschitz and nonnegative. If $N \leq 10$, then u must be constant.

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<u>Remark :</u> If u is not bounded below then the latter does not hold

Symmetry results for stable solutions

Theorem (Work in progress)

Let $N \geq 2$ and $u \in C^2(\mathbb{R}^N)$ be a stable solution of

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $f \in C^1(\mathbb{R})$. Assume that

$$\int_{B(0,R)} |\nabla u|^2 = O(R^2 \ln R) \quad \text{as } R \to +\infty. \tag{5}$$

Then, either

1- *u is constant,*

or,

2- *u is a function of x_N (up to a rotation) and monotone in x_N.*

Symmetry results for stable solutions

Theorem (Work in progress)

Let $2 \leq N \leq 4$ *and* $u \in C^2(\mathbb{R}^N)$ *a stable solution of*

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N.$$

where $f \in C^1(\mathbb{R})$. *Assume that :*

H1- $\exists \zeta \in \mathbb{R}$ *such that* $f(t) \geq 0$ *on* $(-\infty, \zeta]$ *and* $f(t) \leq 0$ *on* $(\zeta, +\infty)$,

H2- $|u(x)| = O(|x|^{\frac{4-N}{2}} \ln^{1/2}|x|)$ *as* $|x| \to +\infty$.

Then there exists $u_0 : \mathbb{R} \to \mathbb{R}$ *such that, up to a rotation,*

$$u(x) = u_0(x_N) \quad \text{for any } x \in \mathbb{R}^N.$$

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   (ex : $u(x) = x_2 \sin(x_1)$) .

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1. Note that there is no assumption on the sign of u.
2. u is not necessary bounded below.
3. The stability of the solution is necessary.
   (ex : $u(x) = x_2 \sin(x_1)$) .
4. Consider $u(x) = x_1 x_2$ then u satisfies $-\Delta u = 0$ in $\mathbb{R}^N$.
   However, u is not one-dimensional.

Symmetry results for stable solutions

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Let $2 \leq N \leq 4$ *and* $u \in C^2(\overline{\mathbb{R}^N_+})$ *a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial x_N} = 0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

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where $f \in Lip(\mathbb{R})$ *is non-negative and non-decreasing. Assume that :*

$$u(x) = o(|x|^{\frac{4-N}{2}} \ln^{1/2}|x|) \textit{ as } |x| \to +\infty.$$

Then there exists $u_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *an increasing function such that, up to a rotation,*

$$u(x) = u_0(x_N) \quad \textit{for any } x \in \mathbb{R}^N_+.$$

Dimensions $2 \leq N \leq 11$

Theorem (B., A. Farina, 2025)

Let Ω be an epigraph defined by a function g Lipschitz continuous bounded from below. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded classical solution to

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Assume that $f \in C^1([0, +\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$, then $u \equiv 0$ and $f(0) = 0$.

Remarks :

- This Theorem holds even if g is not Lipschitz continuous : see the following examples :
  1. g is coercive (i.e $\lim_{|x| \to +\infty} g(x) = +\infty$)

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  2. $g(x_1) = e^{x_1}$, $g(x_1, \ldots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1} \sin(jx_j)$.

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- If f is not positive, then the Theorem is false. See $u(x) = \sin^2(x_N)$.

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- If f is not positive, then the Theorem is false. See $u(x) = \sin^2(x_N)$.
- The previous theorem remains true even for $N \geq 12$, if we add an assumption about the behaviour of f at the origin.

Theorem (B., A.Farina, 2025)

Assume $N \geq 12$ and let Ω be an epigraph defined by a function g Lipschitz continuous bounded from below. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded classical solution to

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u \geq 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{6}$$

Assume that $f \in C^1([0,+\infty))$, $f(t) > 0$ for $t > 0$ and $\liminf_{t\to 0^+} \frac{f(t)}{t^s} > 0$, *for some $s \in \left[0, \frac{N-3}{N-5}\right)$.*
Then $u \equiv 0$ and $f(0) = 0$.

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- If $f(0) > 0$ then there exists no solution to the previous Theorem.
- The aim of this theorem is to study the case $f(t) = t^p$ $(p > 1)$. However, here, we are limited on the choice of p.Indeed, we must have

$$1 < p \leq s < \frac{N-3}{N-5}.$$

Theorem (B., A.Farina, 2025)

Assume $N \geq 12$ *and let* Ω *be an epigraph defined by a function* g *Lipschitz continuous bounded from below. Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *be a bounded classical solution to*

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Assume that

$$1 < p \leq p_c(N-1) = \frac{(N-3)^2 - 4(N-1) + 8\sqrt{N-2}}{(N-3)(N-11)}.$$

Then $u \equiv 0$.

Remarks :

- For $N \geq 12$, we have

$$\frac{N-3}{N-5} < \frac{(N-3)^2 - 4(N-1) + 8\sqrt{N-2}}{(N-3)(N-11)}.$$

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Example : For $N = 12$, $\frac{N-3}{N-5} = 1.28$ and $p_c(N-1) = 6.92$.

Remarks :

- For $N \geq 12$, we have

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Example : For $N = 12$, $\frac{N-3}{N-5} = 1.28$ and $p_c(N-1) = 6.92$.

- This Theorem works for others non linearities f.It must satisfies

$$\begin{cases} f \in C^1([0,+\infty)) \cap C^2((0,+\infty)), \quad f(0) = 0, \\ f > 0, \ \textit{nondecreasing and convex in } (0,+\infty) \\ s.t. \quad \lim_{u \to 0^+} \frac{f'(u)^2}{f(u)f''(u)} := q_0 \in [0,+\infty] \end{cases} \tag{7}$$


Thank You
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Theorem (B., A.Farina, 2025)

Assume $N \geq 12$ *and let* Ω *be an epigraph defined by a function* $g \in \mathcal{G}$*. Also suppose that* Ω *is bounded from below and satisfies a uniform exterior cone condition.*
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *be a bounded distributional solution to* (6) *where* f *satisfies* (7)*.*
Suppose that p_0*, the conjugate exponent of* q_0*, satisfies*

$$1 \leq p_0 < p_c(N-1), \tag{8}$$

where p_c *is the Jospeh-Lundgren stability exponent given by*

$$p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

Then $u \equiv 0$*.*

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

Suppose that there exists $\delta \in (0, \frac{\tilde{t}}{2})$ in such a way that

$$\forall k > 0 \quad \exists \varepsilon_k \in (0, \frac{1}{k}) \quad \exists x^k \in \overline{\Sigma^g_{\delta,\tilde{t}-\delta}} \quad \text{such that } u(x^k) > u_{\tilde{t}+\varepsilon_k}(x^k).$$

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$x^k_N \in [\delta, \tilde{t} - \delta]$, thus $x^k_N \to x_\infty \in [\delta, \tilde{t} - \delta]$.

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$$\forall k > 0 \quad \exists \varepsilon_k \in (0, \frac{1}{k}) \quad \exists x^k \in \overline{\Sigma^g_{\delta, \tilde{t}-\delta}} \quad \text{such that } u(x^k) > u_{\tilde{t}+\varepsilon_k}(x^k).$$

$x^k_N \in [\delta, \tilde{t} - \delta]$, thus $x^k_N \to x_\infty \in [\delta, \tilde{t} - \delta]$. We fix

$$u_k(x) = u(x' + (x^k)', x_N)$$

where $x = (x', x_N) \in \Omega^k := \{(x', x_N) \in \mathbb{R}^N, x_N > g_k(x')\}$ and

$$g_k(x') = g(x' + (x^k)').$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

We have

$$\begin{cases} -\Delta u_k = f(u_k) & \text{in} \quad \Omega^k \\ u_k > 0 & \text{in} \quad \Omega^k \\ u_k = 0 & \text{on} \quad \partial\Omega^k \\ u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k) & \\ u_k(x) \leq u_{k,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_k} \end{cases}$$

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We can show that there exists $g_\infty \in C^0(\mathbb{R}^{N-1})$ such that

$$g_k \to g_\infty \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^{N-1}).$$

We denote by Ω^∞ its epigraph.

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

We have

$$\begin{cases} -\Delta u_k = f(u_k) & \text{in} \quad \Omega^k \\ u_k > 0 & \text{in} \quad \Omega^k \\ u_k = 0 & \text{on} \quad \partial\Omega^k \\ u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k) & \\ u_k(x) \leq u_{k,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_k} \end{cases}$$

We can show that there exists $g_\infty \in C^0(\mathbb{R}^{N-1})$ such that

$$g_k \to g_\infty \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^{N-1}).$$

We denote by Ω^∞ its epigraph.
And there exists $u_\infty \in C^2(\Omega^\infty)$ such that

$$u_k \to u_\infty \quad \text{in } C^0_{\text{loc}}(\Omega^\infty).$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

Moreover u_∞ solves

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ u_\infty \geqslant 0 & \text{in} \quad \Omega^\infty, \\ u_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \\ u_\infty(0', x_\infty) > u_{\infty,\tilde{t}}(0', x_\infty), & \\ u_\infty(x) \leq u_{\infty,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}. \end{cases}$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

Moreover u_∞ solves

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ u_\infty \geqslant 0 & \text{in} \quad \Omega^\infty, \\ u_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \\ u_\infty(0', x_\infty) > u_{\infty,\tilde{t}}(0', x_\infty), & \\ u_\infty(x) \leq u_{\infty,\tilde{t}}(x) & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}. \end{cases}$$

We have

$$-\Delta u_\infty + L_f u_\infty \geq 0 \quad \text{in } \Omega$$

thus by the maximum principle

$$\text{either } u_\infty \equiv 0 \qquad \text{or either } u_\infty > 0.$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

If we fix $w = u_{\infty,\tilde{t}} - u_\infty$ then we have

$$\begin{cases} -\Delta w + L_f w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w(0', x_\infty) = 0 & \end{cases}$$

Proof of proposition ($\tilde{t} < +\infty$) with $f(0) > 0$

If we fix $w = u_{\infty,\tilde{t}} - u_{\infty}$ then we have

$$\begin{cases} -\Delta w + L_f w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w \geq 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty}, \\ w(0', x_\infty) = 0 & \end{cases}$$

Therefore, by the maximum principle $w \equiv 0$ in connected componant of $\Sigma_{\tilde{t}}^{g_\infty}$ which contains $(0', x_\infty)$.

Theorem (Hopf's lemma)

Let $\Omega \subset \mathbb{R}^N$ *be a domain and* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and* $c \in L^\infty(\Omega)$ *such that*

$$\begin{cases} -\Delta u + cu \geqslant 0 & in \quad \Omega \\ u \geqslant 0 & in \quad \Omega \end{cases}$$

Then

1. *If there exists* $x_0 \in \Omega$ *such that* $u(x_0) = 0$ *then*

$$u \equiv 0 \quad in \quad \Omega.$$

2. *Ifnot*

$$u > 0 \quad in \quad \Omega,$$

and if $y_0 \in \partial\Omega$, $u(y_0) = 0$, *and* Ω *satisfies the interior ball condition at* y_0 *then*

$$\frac{\partial u}{\partial \nu}(y_0) < 0.$$

where ν *is the exterior unit normal to* Ω *at* y_0.