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par

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Qualitative properties and geometric aspects of solutions to nonlinear partial differential equations in unbounded domains

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Le manuscrit que vous vous apprêtez à lire est l'aboutissement de 3 ans de thèse. C'est bien sûr la conclusion de 3 ans (et 2 mois) de travail, mais pas que. Ce manuscrit représente également des rencontres avec des gens formidables, des bons et (parfois) des mauvais moments passés avec ma famille ou mes amis. À travers les quelques mots qui suivent, je veux remercier toutes les personnes qui m'ont accompagné durant ces 3 ans et même avant.[1]

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[2]Ils feraient passer Rantanplan pour un génie

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Notations:

$\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\}$, the open upper half-space of $\mathbb{R}^N$.

$|\cdot|$: the Euclidean norm.

$B(x,R)$: the Euclidean N-dimensional open ball of center x and radius $R > 0$.

$B'(x',R)$: the Euclidean $N-1$-dimensional open ball of center x' and radius $R > 0$.

$B_R := B(0,R)$ and $B'_R := B'(0',R)$, where $0 = (0',0) \in \mathbb{R}^{N-1} \times \mathbb{R}$ is the origin of $\mathbb{R}^N$.

$\mathcal{H}^{N-1}$: the $N-1$-dimensional Hausdorff measure.

$UC(X)$: the set of uniformly continuous functions on X.

$Lip(X)$: the set of globally Lipschitz-continuous functions on X.

$Lip_{loc}(X)$: the set of locally Lipschitz-continuous functions on X.

$C^k(\overline{U})$: the set of functions in $C^k(U)$ all of whose derivatives of order $\leq k$ have continuous (not necessarily bounded) extensions to the closure of the open set U.

$C^{0,\alpha}(\overline{U})$: the vector space of bounded and globally α-Hölder-continuous functions h on the open set U endowed with the norm :

$$\|h\|_{C^{0,\alpha}(\overline{U})} := \|h\|_{L^\infty(U)} + [h]_{C^{0,\alpha}(U)} := \sup_{x\in U} |h(x)| + \sup_{x,y\in U, x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\alpha} \ .$$

$C^{k,\alpha}(\overline{U})$: the vector space of functions in $C^k(U)$ all of whose derivatives of order $\leq k$ belong to $C^{0,\alpha}(\overline{U})$, endowed with the norm :

$$\|h\|_{C^{k,\alpha}(\overline{U})} := \textstyle\sum_{0\leq|\beta|\leq k} \|\partial^\beta h\|_{C^{0,\alpha}(\overline{U})} \ .$$

$\mathcal{D}'(U)$: the space of distributions on the open set U.

$H^1_{loc}(\overline{U}) = \{u : U \mapsto \mathbb{R},\ u$ Lebesgue-mesurable $:\ u \in H^1(U \cap B(0,R)) \quad \forall\, R > 0\}$,
i.e., u is Lebesgue-measurable on the open set U and $u \in H^1(V)$ for any open bounded set $V \subset U$.

$W^{1,\infty}_{loc}(\overline{U}) = \{u : U \mapsto \mathbb{R},\ u$ Lebesgue-mesurable $:\ u \in W^{1,\infty}(U \cap B(0,R)) \quad \forall\, R > 0\}$,
i.e., u is Lebesgue-measurable on the open set U and $u \in W^{1,\infty}(V)$ for any open bounded set $V \subset U$.

Introduction

In this manuscript, we study properties of solutions to the semilinear Poisson equation

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{SPE}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a (locally or globally) Lipschitz-continuous function and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an unbounded open set.

This problem has been received a lot of attention in the last decades and is the origin of several conjectures and open problems. In a series of articles (see [BCN93], [BCN96], [BCN97a], [BCN97b]) H. Berestycki, L.A. Caffarelli and L. Nirenberg study problem (SPE) for different type of domains Ω and conjecture several results.

In [BCN97a], the authors deal with bounded solutions of (SPE) in the half-space and prove the following result:

Theorem 0.0.1 (Berestycki-Caffarelli-Nirenberg). *In the half-space $\Omega = \mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N,\ x_N > 0\}$, with $N = 2$ or 3, let u be a bounded classical solution of* (SPE). *If $N = 2$ and f is locally Lipschitz-continuous, then u is symmetric $(i.e.\ u = u(x_N))$. If $N = 3$, then the same conclusion holds, if one assumes, in addition, that $f(0) \geq 0$ and that f is C^1.*

In order to prove this theorem, the authors show that $f(\sup u) = 0$ and then, apply the following Theorem (see [BCN93]):

Theorem 0.0.2 (Berestycki-Caffarelli-Nirenberg). *Let u be a bounded classical solution to* (SPE) *in the half-space $\mathbb{R}^N_+$, with f locally Lipschitz-continuous that satisfies*

$$f(\sup u) \leq 0.$$

Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$ and u only depends on the variable x_N and $f(\sup u) = 0$.

In view of these results, the authors formulated the following:

Conjecture A. *If there is a bounded solution u to* (SPE) *with $\Omega = \mathbb{R}^N_+$ then $f(\sup u) = 0$ (so Theorem 0.0.2 applies) and consequently u is symmetric and monotone $(i.e.\ \frac{\partial u}{\partial x_N} > 0)$.*

In this respect, A. Farina and E. Valdinoci (see Theorem 1.11 in [FV10]) prove a symmetry result for bounded solutions of (SPE) in $\mathbb{R}^N_+$, with $N \leq 5$ for functions $f \in C^1([0,+\infty))$ satisfying one of the following two conditions:

(a) $f(t) \geq 0$, for any $t \geq 0$,

(b) there exists $\zeta > 0$, such that $f(t) \geq 0$ for any $t \in [0,\zeta]$ and $f(t) \leq 0$ for any $t \in [\zeta,+\infty)$.

L. Dupaigne and A. Farina ([DF22]) extended this result up to dimension 11 (see Theorem 0.4.8 in section 0.4). By exploiting (a) or (b), the authors prove that $f(\sup u) = 0$ and then they conclude by appyling Theorem 0.0.2 above.

In [BCN97a], the authors test Conjecture A by considering $f(u) = u - 1$ and they find that in this case the conjecture is not true. Indeed, they observe that if there exists a bounded solution u to (SPE) with $\Omega = \mathbb{R}^N_+$ and $f(u) = u - 1$ then $\sup u > 2$, but $f(t) > 0$ for any $t \geq 2$. This observation leads them to raise the following conjecture:

Conjecture B. *There is no bounded solution of*

$$\begin{cases} -\Delta u = u - 1 & \text{in} \quad \mathbb{R}^N_+, \\ u > 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

In [BCN97a], H. Berestycki, L.A. Caffarelli and L. Nirenberg prove this conjecture in dimensions 2 and 3 (see Proposition 1.6 in [BCN97a]). In 2016, in [CEG16], C. Cortázar, M. Elgueta and J. Garcia-Melián completely solve Conjecture B, namely, they prove the following result:

Theorem 0.0.3 (Cortázar-Elgueta-Garciá-Melian). *Assume $N \geq 2$. The only classical solution, possibly unbounded, to problem*

$$\begin{cases} -\Delta u = u - 1 & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{0.0.1}$$

is the function

$$u(x) = 1 - \cos(x_N) \quad \text{for any } x \in \mathbb{R}^N_+.$$

This result was first demonstrated by A. Farina and B. Sciunzi in dimension 2 (see [FS16]).

The result above suggests that also the non-negative solutions to the homogeneous Dirichlet problem for semilinear equations has to be taken into account. For this reason, in this thesis, we also study non negative and possibly unbounded solutions to the following problem (see subsection 0.1.4 and section 0.4, where further motivations are also discussed):

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{SPE+}$$

So far, the results and conjectures have been stated in the half-space $\mathbb{R}^N_+$, and a natural question arises: *what happens when the geometry of* Ω *is more complex?*

A natural generalisation of the half-space is provided by the case of an epigraph, i.e., when Ω has the following form:

$$\Omega := \{(x', x_N) \in \mathbb{R}^N, x_N > g(x')\}, \tag{0.0.2}$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous function.

For such domains, M.J. Esteban and P.L. Lions prove the following monotonicity result:

Theorem 0.0.4 (Esteban-Lions). *Let $g : \mathbb{R}^{N-1} \to \mathbb{R}$ be a sufficiently smooth function which is coercive,i.e., satisfying*

$$\lim_{|x'| \to +\infty} g(x') = +\infty, \tag{0.0.3}$$

and let Ω be its epigraph (see (0.0.2)*). Then, if $u \in C^2(\overline{\Omega})$ is a solution to* (SPE) *with $f \in Lip_{loc}([0, +\infty))$ and $f(0) \geq 0$, then*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{in } \Omega.$$

This is the first monotonicity result in epigraphs, and the question is whether the theorem 0.0.4 remains true when the epigraph Ω is no longer necessarily coercive and/or when $f(0) < 0$. In this manuscript, we address this question.

In his seminal paper published in 1971 (see [Ser71]), J. Serrin, in order to answer a question in fluids mechanics posed by R.L. Fosdick, is led to consider the classical solutions of the following overdetermined problem:

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{SOP}$$

where Ω is a smooth bounded domain, η denotes the outward unit normal at $\partial\Omega$ and f is a Lipschitz-continuous function. To solve the problem, J. Serrin, in [Ser71], proves his celebrate result:

Theorem 0.0.5 (Serrin). *Let Ω be a bounded domain whose boundary is of class C^2. Suppose that there exists a function $u \in C^2(\overline{\Omega})$ satisfying* (SOP)*. Then Ω is a ball and u has the following specific form:*

$$u(x) = \frac{b^2 - |x|^2}{2N},$$

where b is the radius of the ball and $|x|$ the denotes distance from its center.

To prove this beautiful result, J.Serrin introduced the PDE community to the celebrated moving planes method (based on Alexandrov's reflexion principle (see [Ale62])).

In [BCN97b], H. Berestycki, L.A. Caffarelli and L. Nirenberg study problem (SOP) when Ω is a smooth epigraph (see 0.0.2) and f is an Allen-Cahn type function (see Definition 0.1.1). The authors employ the sliding method to prove the following rigidity result :

Theorem 0.0.6 (**Berestycki-Caffarelli-Nirenberg**). *Let f be an Allen-Cahn type function. Let $g : \mathbb{R}^{N-1} \to \mathbb{R}$ be a globally-Lipschitz continuous function of class C^2 which satisfies the following condition:*

$$\text{for any } \tau \in \mathbb{R}^{N-1} \qquad \lim_{x \in \mathbb{R}^{N-1}, |x| \to \infty} g(x+\tau) - g(x) = 0, \tag{0.0.4}$$

and let Ω be its epigraph. If there exists a bounded solution u to the problem (SOP) *then Ω is an upper half-space and u is symmetric and monotone.*

Following this result, the authors of [BCN97b] formulate the following question:

Conjecture C. *Assuming that Ω is a smooth domain with Ω^c connected and that there exists a bounded positive solution of* (SOP) *for some Lipschitz-continuous function f, then Ω is either a half-space, a ball, the complement of a ball, or a circular-cylinder-type domain: $\mathbb{R}^j \times B$ with B a ball.*

This question is known as the conjecture of Berestycki-Caffarelli-Nirenberg. Motivated by Theorem 0.0.6, in this thesis we address Conjecture C in the special case where Ω is an epigraph, which leads to the following variant of Conjecture C:

Conjecture D. *If f is a globally Lipschitz-continuous function and the overdetermined problem* (SOP) *admits a smooth and bounded classical solution on a smooth epigraph Ω, then Ω must be an affine half-space.*

All these conjectures and previous results led us to study geometric properties of solutions to (SPE) in various unbounded domains. In the remainder of this introduction, we present in a unified manner, the results obtained during this thesis which we compare with existing works in the literature. All these results are contained in three original research articles that are the subject of chapters 1, 2 and 3 (see [BFS25], [BF25b], [BF25a]).

Specifically:

1) In section 0.1, we introduce our new monotonicity results. In the first subsection, we describe new monotonicity results for possibly unbounded solutions to (SPE) in general epigraphs bounded from below and for any locally or globally Lipschitz-continuous function satisfying $f(0) \geq 0$ (see subsection 0.1.1). The second subsection is devoted to our monotonicity results that hold without any restriction on the sign of $f(0)$ (see subsection 0.1.2). These results are established for possibly unbounded solutions to (SPE) or (SOP) in various unbounded domains. In the third subsection, we present extensions of our monotonicity results to a broad class of merely continuous epigraphs, bounded from below (see subsection 0.1.3). The proof of all these results

is based on some new comparison principles in unbounded domains. We shall present and describe these new principles in the second section below. Finally, in the last subsection, we provide a first applications of our monotonicity results to the classification of bounded solutions of (SPE+) in epigraphs bounded from below (see subsection 0.1.4).

2) In section 0.2, we establish new comparison principles (see subsection 0.2.2) for a large class of unbounded Euclidean subsets, that we call *sets with good sections* (see subsection 0.2.1). These comparison principles are used to initialise and to complete a modified version of the moving plane procedure adapted to the geometry of the epigraph Ω. This is crucial for proving our monotonicity results described in the first section. Finally, our new comparison principles allow us to prove some new results of uniqueness and symmetry for solutions, possibly unbounded and sign-changing, to the homogeneous Dirichlet BVP for the semilinear Poisson equation in fairly general unbounded domains (see subsection 0.2.3).

3) In section 0.3, we prove the validity of Conjecture D in several cases. Specifically, we consider Serrin's overdetermined problem (SOP) for a family of specified Allen-Cahn-type nonlinearities. In this framework, we prove Conjecture D in any dimension $N \geq 2$, for any solution to (SOP) and for a large class of epigraphs bounded from below (see subsection 0.3.1). Finally, we consider solutions to (SOP) in a smooth enough epigraph bounded from below where f is a general locally or globally Lipschitz-continuous function. In this framework, we prove Conjecture D with $N \leq 3$, for possibly unbounded solutions u (see subsection 0.3.2).

4) The last section (see section 0.4) is devoted to present some new classification results for possibly unbounded solutions to (SPE+) in the half-space.

    (i) provide general natural conditions on f guaranteeing that the only classical solutions, bounded or not, to (SPE+) are necessarily one-dimensional;

    (ii) supply sufficiently general assumptions on classical solutions to (SPE+) ensuring their one-dimensional symmetry.

    The flexibility of our approach enabled us to to extend some of the previous results to the case of differential inequalities, even without imposing any boundary conditions.

0.1 Monotonicity results in unbounded domains

In this section, we present our new monotonicity results for solutions u (bounded or not) to (SPE) on unbounded domains included in a half-space, as well as for general (locally or globally) Lipschitz-continuous non-linearities f. In particular, we are interested in the case where Ω is an epigraph bounded from below, i.e.,

$$\Omega := \{(x', x_N) \in \mathbb{R}^N,\, x_N > g(x')\},$$

with $g : \mathbb{R}^{N-1} \to \mathbb{R}$ a continuous function bounded from below.

We recall that, when $f(0) \geq 0$, the maximum principle ensures that a solution to (SPE+) in any domain Ω is either identically equal to zero or positive and that, this property fails when $f(0) < 0$ (see the example given by $f(u) = u - 1$ and Theorem 0.0.3). In view of this observation, we have decided in this section to first present the results in the case $f(0) \geq 0$, and then those that are valid regardless of the sign of $f(0)$. The case $f(0) \geq 0$ is discussed in subsection subsection 0.1.1, while the monotonicity results that hold irrespectively of the sign of $f(0)$ are located in subsection (0.1.2).
Subsection 0.1.3 is devoted to present some extensions of our monotonicity results to a broad class of epigraphs which are merely continuous and bounded from below.
Finally, in the last subsection (see subsection 0.1.4), we describe a first application of our new monotonicity results. This concerns the classification of bounded solutions to (SPE+) in epigraphs bounded from below.
In each subsection, we first present the existing results in the mathematical literature, and then our new results on monotonicity.

0.1.1 Monotonicity results when $f(0) \geq 0$

The case where $f(0) \geq 0$ is the most studied. The first result in this respect is due to E.N. Dancer ([Dan92]). In this paper, the author studies problem (SPE) in the half-space and, thanks to the moving-plane method, he proves the following result:

Theorem 0.1.1 (Dancer). *Assume that f is C^1, with $f(0) > 0$ or both $f(0) = 0$ and $f'(0) \geq 0$, and that u is a bounded solution of* (SPE). *Then $\frac{\partial u}{\partial x_N}(x) > 0$ if $x \in \overline{\mathbb{R}^N_+}$.*

Then, few years later H. Berestycki, L.A. Caffarelli and L. Nirenberg, in [BCN97a], prove the following (see corollary 1.3 in [BCN97a]):

Theorem 0.1.2 (Berestycki-Caffarelli-Nirenberg). *In the half-space $\Omega = \mathbb{R}^N_+$, assume that u is a solution of* (SPE) *where f is a globally Lipschitz-continuous function with $f(0) \geq 0$. Then, the function u satisfies*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{in } \Omega.$$

This result applies to possibly unbounded solutions to (SPE). However, the authors assume that f is a globally Lipschitz-continuous function and therefore Theorem 0.1.2 does not apply to functions such as $f(t) = t^p$ $(p > 1)$ or $f(t) = t - t^3$ (which give respectively the Lane-Emden equation and the Allen-Cahn equation). For this reason, the same authors, in [BCN96], raise the following question:

Open problem 0.1.1. `Does the conclusion of Theorem 0.1.2 still holds for unbounded solutions of` (SPE) `in the case where` f `is merely locally Lipschitz-continuous ?`

The proof of Theorem 0.1.2 relies on the moving plane method. To start this procedure, the authors of [BCN97a] use one of their results established in [BCN96], where they prove that positive solutions to (SPE) in $\mathbb{R}^N_+$ have at most exponential growth on finite strips [3]. A result that crucially uses the global Lipschitz character of f.

In [Far20], A. Farina considers the problem (SPE) in the half-space, where f is only locally Lipschitz-continuous and he gives a first answer to the open problem (0.1.1). Indeed, he proves the following result:

Theorem 0.1.3 (**Farina**). *Assume $N \geq 2$, $\Omega = \mathbb{R}^N_+$, $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution to* (SPE). *Assume that u is bounded on finite strips $\mathbb{R}^{N-1} \times [0, t]$, for any $t > 0$, that is,*

$$\forall t > 0, \ \exists C(t) > 0 \,_{,} \textit{such that } 0 < u \leq C(t) \quad \textit{in } \mathbb{R}^{N-1} \times [0, t].$$

Then u is strictly increasing in the x_N-direction, i.e, $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.

This result applies to any locally Lipschitz-continuous function, in particular to $f(t) = t - t^3$ and $f(t) = t^p$, but it requires u to be bounded on finite strips.

All the previous results are proven by crucially exploiting the fact that $\Omega = \mathbb{R}^N_+$. Therefore, it seems to be interesting to know what happens when Ω is an unbounded domain different from the half-space.

In this context, there are few results and the first contribution in this direction is due to M.J. Esteban and P.L. Lions who, in 1982, considered problem (SPE) for a particular class of unbounded domains (see [EL82]).

Theorem 0.1.4 (**Esteban-Lions**). *Let Ω be a sufficiently smooth domain satisfying the following conditions:*

(i) If $x = (x_1, \cdots, x_N) \in \Omega$ then $y = (y_1, \cdots, y_N) \in \Omega$, for any $y_N \geq x_N$,

(ii) $m = \inf\{x_N |$ there exists $x' \in \mathbb{R}^{N-1}$ such that $x = (x', x_N) \in \Omega\} > -\infty$,

(iii) for each $\lambda > 0$, the set $\Sigma(\lambda) := \{x \in \Omega | \, x_N < \lambda\}$ is bounded.

Then, if $u \in C^2(\overline{\Omega})$ is a solution to (SPE) *with $f \in Lip_{loc}([0, +\infty))$ and $f(0) \geq 0$, then*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{in } \Omega.$$

It is easy to check that any coercive epigraph satisfies the assumptions (i), (ii) and (iii).

Later on, in [BCN97b], H. Berestycki, L.A. Caffarelli and L. Nirenberg deal with the problem (SPE) in a smooth globally Lipschitz continuous epigraph where f is an Allen-Cahn type function, that is:

[3] i.e. u satisfies the following property : for any $b > 0$ there are $\alpha, \beta > 0$, depending on b, such that

$$u(x', x_N) \leq \alpha e^{\beta |x'|} \quad \text{for any } (x', x_N) \in \mathbb{R}^{N-1} \times [0, b].$$

Definition 0.1.1. $f:[0,+\infty)\to\mathbb{R}$ is an Allen-Cahn type function if f is a locally Lipschitz-continuous function and satisfies the following assertions:

$$
\begin{cases}
 & \exists\,\mu>\delta_2>\delta_1>0\ : \\
\text{(I)} & f>0 \quad \text{in } (0,\mu), \qquad f\le 0 \quad \text{in } [\mu,+\infty), \\
\text{(II)} & f(t)\ge\delta_1 t \quad \text{for any } t\in(0,\delta_1), \qquad f \text{ is non-increasing in } (\delta_2,\mu).
\end{cases}
\tag{0.1.1}
$$

The prototype of the function satisfying the previous definition is $f(t)=t-t^3$, which gives the Allen-Cahn equation $-\Delta u=u-u^3$ (see [AC79]), hence the name 'Allen-Cahn type function'. For such functions, the authors prove the following Theorem:

Theorem 0.1.5 (Berestycki-Caffarelli-Nirenberg). *Let Ω be a globally Lipschitz-continuous epigraph and let $f:[0,+\infty)\to\mathbb{R}$ be an Allen-Cahn type function. If $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ is a bounded solution of* (SPE)*, then u is strictly increasing in the x_N-direction, i.e.,*

$$\frac{\partial u}{\partial x_N}>0 \quad \text{in } \Omega.$$

This result applies to any globally Lipschitz-continuous epigraphs, however the proof is crucially based on the particular shape of f, which is of Allen-Cahn type, as well as on the boundedness of u.

To prove this result, the authors use the sliding method (see Lemma 3.1 in [BCN97b]) which consists in showing that if two functions u and z are ordered in a ball B, with $\overline{B}\subset\Omega$, and they satisfy:

$$
\begin{cases}
-\Delta u\ge f(u) & \text{in} & \Omega, \\
-\Delta z\le f(z) & \text{where } z>0 \text{ in } B, & \\
z\le u & \text{in} & B, \\
z\le 0 & \text{on} & \partial B,
\end{cases}
$$

then, for any continuous one-parameter family of Euclidean motions (i.e., translations and rotations) $A(t)$ for $0\le t\le T$ with $A(0)=Id$ and $A(t)\overline{B}\subset D$ for any t, we have for all $t\in[0,T]$:

$$z_t(x):=z(A(t)^{-1}x)<u(x) \text{ in } B_t:=A(t)B.$$

We are now in position to state our new monotonicity results in epigraphs bounded from below and for globally or locally Lipschitz-continuous functions f. Let us begin with the case of uniformly continuous epigraphs:

Theorem 0.1.6 (B.-Farina-Sciunzi). *Let Ω be a uniformly continuous epigraph bounded from below. Assume $f\in Lip_{loc}([0,+\infty))$ with*

$$\liminf_{t\to 0^+}\frac{f(t)}{t}>0, \tag{0.1.2}$$

and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *be a distributional solution to* (SPE) *which is bounded and uniformly continuous on finite strips.*[4] *Then* u *is strictly increasing in the* x_N*-direction, i.e.,* $\frac{\partial u}{\partial x_N} > 0$ *in* Ω.[5]
If f *is globally Lipschitz-continuous, the same conclusion holds true only under the sole assumption that* u *is uniformly continuous on finite strips.*

The epigraphs covered by the previous result are very general, as shown by the two-dimensional epigraph of the Weierstrass-type function

$$g_{b,\alpha}(x) := \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n \pi x),$$

where $b > 1$ is an integer and $\alpha \in (0, 1)$. The function $g_{b,\alpha}$ is uniformly continuous, bounded and nowhere differentiable, hence it is not even locally Lipschitz-continuous.

The proof of Theorem 0.1.6 relies in part on the moving plane method adapted to the geometry of epigraphs. To initialize this method, we must require the epigraphs to be bounded from below. If we further assume that the epigraph satisfies a weak regularity assumption, we can prove the monotonicity result for any (locally or globally) Lipschitz-continuous function f satisfying $f(0) \geq 0$ (and this by also weakening the assumptions on u).

Theorem 0.1.7 (**B.-Farina-Sciunzi**). *Let* Ω *be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition.*
Assume $f \in Lip([0, +\infty))$ *with* $f(0) \geq 0$ *and let* $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *be a distributional solution to* (SPE) *with at most exponential growth on finite strips, i.e., for any* $R > 0$, *there are positive numbers* $A = A(R), B = B(R)$ *such that*

$$u(x) \leq Ae^{B|x|} \qquad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

Then u *is strictly increasing in the* x_N*-direction, i.e.,* $\frac{\partial u}{\partial x_N} > 0$ *in* Ω.

Let us recall the uniform exterior cone condition:

Definition 0.1.1 (**Uniform exterior cone condition**). An open set $\omega \subset \mathbb{R}^N$ (not necessarily an epigraph) satisfies a *uniform exterior cone condition* if for any $x_0 \in \partial\omega$ there exists a finite right circular cone V_{x_0}, with vertex x_0, such that $\overline{\omega} \cap V_{x_0} = \{x_0\}$ and the cones V_{x_0} are all congruent to some fixed cone V. The cone V is called the *reference cone.*

It is easy to see that all globally Lipschitz-continuous epigraphs satisfy this property and that, in this case, the reference cone depends on the Lipschitz constant of the function defining the epigraph. However, the converse is not true, as shown by the following examples. The epigraph defined by function $x \to e^X$ satisfies the uniform exterior cone condition and is not uniformly continuous. The epigraph defined by the function g below, is $\frac{1}{2}$-Hölder-continuous satisfies the uniform exterior cone condition but it is not globally Lipschitz-continuous (see figure (1)).

[4]i.e, for any $R > 0$, u is bounded and uniformly continuous on the strip $\Omega \cap \{x_N < R\}$
[5]Continuous distributional solutions of $-\Delta u = f(u)$ in Ω belong to $C^2(\Omega)$, by standard elliptic theory.

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, -4], \\ \sqrt{4-(x+2)^2} & \text{if} \quad x \in [-4, 0], \\ \sqrt{4-(x-2)^2} & \text{if} \quad x \in [0, 2], \\ 2 & \text{if} \quad x \in [2, +\infty), \end{cases}$$

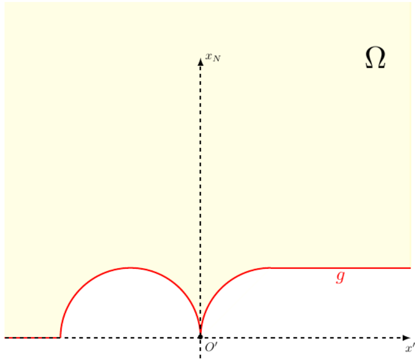

Figure 1: g

Note that Theorem 0.1.7 recovers Theorem 0.1.2 of H. Berestycki, L.A. Caffarelli and L. Nirenberg when $\Omega = \mathbb{R}^N_+$, since in this case, as already discussed above, the solution u has at most exponential growth in finite strips (see [BCN96]).

We also study solutions to (SPE) in an uniformly continuous epigraph bounded from below when f is a locally Lipschitz-continuous function and we prove the following result:

Theorem 0.1.8 (B.-Farina-Sciunzi). *Let Ω be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition.*
Assume $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE) *which is bounded on finite strips, i.e., for any $R > 0$,*

$$\sup_{\Omega \cap \{x_N < R\}} u < +\infty. \tag{0.1.3}$$

Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

When $\Omega = \mathbb{R}^N_+$, Theorem 0.1.8 recovers Theorem 0.1.3 of A. Farina.

We also note that Theorem 0.1.8 and Theorem 0.1.7 cover the case of uniformly continuous epigraphs bounded from below, which are not necessarily globally Lipschitz-continuous, as illustrates by function g defined below (see figure 2):

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, -4], \\ \sqrt{4-(x+2)^2} & \text{if} \quad x \in [-4, 0], \\ \sqrt{4-(x-2)^2} & \text{if} \quad x \in [0, 2], \\ 2 & \text{if} \quad x \in [2, 6], \\ x-4 & \text{if} \quad x \in [6, +\infty). \end{cases}$$

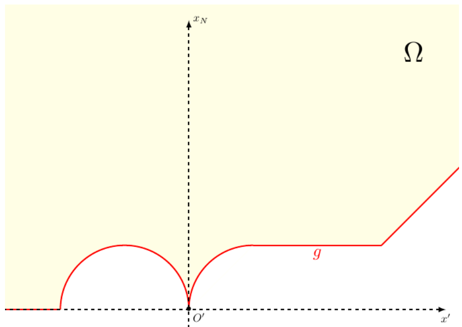

Figure 2: g

Note that g is uniformly continuous, bounded from below, but not locally Lipschitz-continuous. Moreover, it is easily seen that it satisfy a uniform exterior sphere condition (of radius $\frac{1}{2}$). Hence, it also satisfy a uniform exterior cone condition.

We now present some special cases of the previous results. Even in this weaker form, they are new.

Corollary 0.1.1 (B.-Farina-Sciunzi). *Let Ω be a uniformly continuous epigraph bounded from below. Assume $f \in Lip_{loc}([0, +\infty))$ with*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0, \tag{0.1.4}$$

and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of (SPE) *such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Corollary 0.1.2 (B.-Farina-Sciunzi). *Let Ω be a globally Lipschitz-continuous epigraph bounded from below. Assume $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of* (SPE) *such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Corollary 0.1.3 (B.-Farina-Sciunzi). *Assume $\alpha \in (0, 1)$ and let Ω be a globally Lipschitz-continuous epigraph bounded from below with $g \in C^{1,\alpha}_{loc}(\mathbb{R}^{N-1})$. Assume $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of* (SPE) *which is bounded on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

0.1.2 Monotonicity results independent of the sign of $f(0)$

In this subsection, we present monotonicity results which hold regardless the sign of $f(0)$. Because of the lack of the strong maximum principle and the Hopf's Lemma when $f(0) < 0$, very few results are known.

Moreover, when $f(0) < 0$, we can have non-negative solutions to (SPE) that are not increasing in the x_N direction, and sometimes they are the only non-negative solutions, as shown by the case of $f(u) = u - 1$ (see [FS16], [BCN97a], [CEG16], [FS13]).

The first result that we quote is a monotonicity and symmetry result due to H. Berestycki, L.A. Caffarelli and L. Nirenberg. Indeed, in [BCN93], they prove the following:

Theorem 0.1.9 (Berestycki-Caffarelli-Nirenberg). *Let $u \in C^2(\mathbb{R}^N_+)$ be a bounded solution to* (SPE) *in the half-space, with f a locally Lipschitz-continuous function which satisfies*

$$f(\sup u) \leq 0.$$

Then u is strictly increasing in the x_N-direction, i.e, $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$ and u only depends on the variable x_N and $f(\sup u) = 0$.

The boundedness assumption on u is sharp, for instance, the unbounded, increasing function $u(x_1, \cdots, x_N) = x_N e^{x_1}$ solves (SPE) in the half space with $f(u) = -u$.

In this context, H. Berestycki, L.A. Caffarelli and L. Nirenberg, in [BCN97a], prove that the monotonicity result in Theorem 0.1.9 remains true in $\mathbb{R}^2_+$ for globally Lipschitz-continuous function f, without any assumptions on u, precisely they show the following result:

Theorem 0.1.10 (**Berestycki-Caffarelli-Nirenberg**). *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to* (SPE) *in $\mathbb{R}^2_+$, with $f \in Lip([0, +\infty))$. Then,*

$$\frac{\partial u}{\partial x_2} > 0 \quad in \quad \mathbb{R}^2_+.$$

In [FS17], A. Farina and B. Sciunzi prove the same monotonicity result for any locally lipschitz continuous function f. In particular, their result provides a positive answer to the open problem 0.1.1 and it recovers Theorem 0.1.10 which holds for globally Lipschitz-continuous function. More precisely, they show the following Theorem:

Theorem 0.1.11 (**Farina-Sciunzi**). *Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to* (SPE) *in $\mathbb{R}^2_+$.*
Then

$$\frac{\partial u}{\partial x_2} > 0 \quad in \quad \mathbb{R}^2_+.$$

The proof of Theorem 0.1.11 is based on a variant of the moving plane method which is called *rotating line method.* This method consists in comparing solution u of (SPE) and its symmetric with respect to a straight line passing throught a point $(x_0, s) \in \mathbb{R}^2_+$ and of slope $\tan(\theta)$ with $\theta \in (0, \frac{\pi}{2})$. By doing this construction, the triangle formed by this line and the axes (Ox) and (Oy) is bounded, the fact that we are in dimension 2 comes into play here. The authors can then use the maximum principle for bounded domains and for domains of small volume (see [FS16], [FS17]) to complete the procedure.

This construction is purely two-dimensional, and it does not seem possible to reproduce it in higher dimensions. So, the case $N \geq 3$ remains an open problem.

When considering unbounded domains Ω different from $\mathbb{R}^N_+$, very few results are known. In this respect, we quote a result of H. Berestycki, L.A. Caffarelli and L. Nirenberg (see [BCN97b]) extending Theorem 0.1.4 of M.J. Esteban and P.L. Lions (see [EL82]).

Theorem 0.1.12 (**Berestycki-Caffarelli-Nirenberg**). *Suppose that $\Omega \subset \mathbb{R}^N$ is a smooth locally Lipschitz-continuous coercive epigraph. Let u be a solution of* (SPE) *where $f \in Lip_{loc}([0, +\infty))$. Then*

$$\frac{\partial u}{\partial x_N} > 0 \quad in\ \Omega.$$

We also study the case of coercive epigraphs. As a consequence of the method develop in our first article (see chapter 1), we prove the following result:

Theorem 0.1.13 (B.-Farina-Sciunzi). *Let Ω be a coercive continuous epigraph. Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (SPE). *Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

There is no assumption concerning the smoothness of Ω and the sign of $f(0)$, hence, we recover Theorem 0.1.4 of M.J. Esteban and P.L. Lions and Theorem 0.1.12 of H. Berestycki, L.A. Caffarelli and L. Nirenberg.

When f is a non-increasing continuous function, we also provide a monotonicity result for solutions (bounded or not) to (SPE) in domains Ω which are not necessarily epigraphs but which satisfy the following property:

$\mathcal{R}$: Ω `contains the reflection (with respect to the hyperplane` $\{x_N = \theta\}$`) of all strips` $\Omega \cap \{x_N < \theta\}$.

More precisely, we have the following result:

Theorem 0.1.14 (B.-Farina-Sciunzi). *Let Ω be any unbounded domain included in a half space, bounded from below which satisfies property $\mathcal{R}$ and let $f : [0, \infty) \mapsto \mathbb{R}$ be any non-increasing function. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (SPE) *with subexponential growth on finite strips* [6].
Then u is non-decreasing, i.e., $\frac{\partial u}{\partial x_N} \geq 0$ in Ω.[7]
Moreover, if $f \in Lip_{loc}$, then u is strictly increasing, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

This result applies to very general unbounded domains (which are not necessarily epigraphs), as $\Omega_4 = \bigcup_{k \in \mathbb{Z}} \{x \in \mathbb{R}^2 \, : \, |x_1 - (3 + 4k)| < 1 \, , \; x_2 > -1\} \bigcup \{x \in \mathbb{R}^2 \, : \, x_2 > 0\}$ (see figure (3))(in fact, Ω_4 is never an epigraph, that is, there is no unit vector ν of $\mathbb{R}^2$ that allows it to be represented as an epigraph with respect to ν), or the open orthant $\Omega_5 = \{x \in \mathbb{R}^N \, : \, x_1 > 0, \dots, x_N > 0\}$ which satisfies property $\mathcal{R}$, but it is not an epigraph with respect to e_N.

[6] i.e., for any $R > 0$,

$$\limsup_{\substack{|x| \to \infty, \\ x \in \Omega \cap \{x_N < R\}}} \frac{\ln u(x)}{|x|} \leq 0.$$

[7] Note that $u \in C^1(\Omega)$, since $f(u) \in L^\infty_{loc}(\Omega)$.

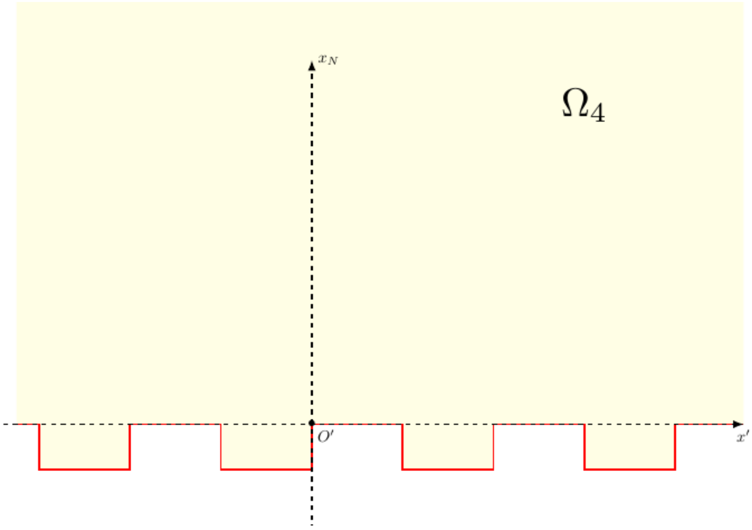

Figure 3: Ω_4

We see that the above result actually caracterizes the euclidean proper domains for which the homogeneous Dirichlet BVP (SPE) admits a monotone solution. This can be formulated in the following equivalent way : Problem (SPE) with f non-increasing, admits a monotone solution on a domain $\Omega \subsetneq \mathbb{R}^N$, if and only if, Ω is contained in an affine half-space (whose inner normal is denoted by ν) and, for any $x \in \Omega$ the open half-line $\{x + t\nu,\, t > 0\}$ is contained in Ω. That is, if and only if, Ω is contained in an affine half-space (whose inner normal is denoted by ν) and Ω is ν-invariant.

So far, we have seen that unless we make strong assumptions about the domain Ω or about the nonlinearity f, we cannot establish results independent of the sign of $f(0)$.

However, when we consider possibly unbounded solutions to (SOP), we can prove monotonicity result for a broad class of epigraphs unbounded from below, regardless the sign of $f(0)$. This is the content of Theorem 0.1.1 below. To this end, we introduce the following class of functions:

Definition 0.1.2. Assume $N \geq 2$ and $\gamma \in (0,1]$. We denote by $\mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ the set of functions $h \in C^1(\mathbb{R}^{N-1})$ such that ∇h is bounded and globally γ-Hölder continuous, i.e., such that

$$\begin{aligned}\|\nabla h\|_{C^{0,\gamma}(\mathbb{R}^{N-1})} &:= \|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})} + [\nabla h]_{C^{0,\gamma}(\mathbb{R}^{N-1})} \\ &:= \sup_{x\in\mathbb{R}^{N-1}} |\nabla h(x)| + \sup_{x,y\in\mathbb{R}^{N-1}, x\neq y} \frac{|\nabla h(x) - \nabla h(y)|}{|x-y|^\gamma} < +\infty.\end{aligned}$$

For epigraphs defined by this type of functions, we prove the following result:

Théorème 0.1.1 (B.-Farina). *Assume $N \geq 2$, $\gamma \in (0,1]$. Let Ω be an epigraph bounded from below and defined by a function $g \in \mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ and let $f \in Lip_{loc}(\mathbb{R})$.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (SOP) *with $\mathfrak{c} \neq 0$ and such that ∇u is bounded on*

finite strips.[8] *Then, u is monotone, i.e.*

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega.$$

In particular, the above Theorem applies to bounded solutions to (SOP). Indeed, bounded solutions to (SOP) also have bounded gradient (see Corollary 2.5.2), but the converse is not true, as shown, by the harmonic function $u(x) = \alpha x_N$ ($\alpha \geq 0$).

0.1.3 Extensions to merely continuous epigraphs bounded from below

We notice that we can extend our monotonicity results to a larger class of epigraphs bounded from below. Indeed, during our proofs, we are led to consider the sequence of translation $g_k(\cdot) = g(\cdot + (x^k)')$ and to study the convergence of this sequence. This is why, we define a new class of functions where the convergence of a such sequence is assumed:

Definition 0.1.3. Assume $N \geq 2$. We say that a continuous function $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ belongs to the class $\mathcal{G}$, if it satisfies the following compactness property:

($\mathcal{P}$) *Any sequence* (g_k) *of translations of* g*, which is bounded at some fixed point of* $\mathbb{R}^{N-1}$*, admits a subsequence converging uniformly on every compact sets of* $\mathbb{R}^{N-1}$*.*

Before state the new monotonicity results, let us show how large the class $\mathcal{G}$ is. Hereafter, we provide a wide (but non-exhaustive) list of members of $\mathcal{G}$.

1. Uniformly continuous functions on $\mathbb{R}^{N-1}$ belong to $\mathcal{G}$.

2. Coercive continuous functions on $\mathbb{R}^{N-1}$ belong to $\mathcal{G}$.

   In particular, any continuous function on $\mathbb{R}^{N-1}$ such that $\lim_{|x| \mapsto \infty} g(x) \in (-\infty, +\infty]$ belongs to $\mathcal{G}$.

3. Let us denote by $\mathsf{G}(\mathbb{R}^{N-1})$ the set of continuous functions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ enjoying the following property : *there exists a* **continuous bijection** $\phi : \mathbb{R} \to \mathbb{R}$ *such that* $\phi \circ g \in \mathcal{G}$.
   It is immediate to check that $\mathsf{G}(\mathbb{R}^{N-1}) \subset \mathcal{G}$.Moreover, the family $\mathsf{G}(\mathbb{R}^{N-1})$ strictly contains the one of *uniformly continuous functions* on $\mathbb{R}^{N-1}$ and the one of *coercive continuous functions* on $\mathbb{R}^{N-1}$, as shown by the next examples.

   (3a) For instance, the functions $g_1(x_1) = e^{x_1}$ if $N = 2$, and $g(x) = e^{x_1 + \sum_{j=2}^{N-1} \cos^j(x_j)}$ if $N \geq 3$, are neither uniformly continuous nor coercive on $\mathbb{R}^{N-1}$. Nevertheless, they belong to the class $\mathsf{G}(\mathbb{R}^{N-1})$ (consider, for instance, the function

$$\phi(t) = \begin{cases} t & \text{if } \ t \leq 0, \\ \log(t+1) & \text{if } \ t > 0, \end{cases}$$

[8] i.e., for any $R > 0$,

$$\sup_{\Omega \cap \{x_N < R\}} |\nabla u| < +\infty.$$

in the previous definition and observe that $\phi \circ g \in \mathcal{G}$ is uniformly continuous, hence $\phi \circ g \in \mathcal{G}$). The same argument also proves that $g(x_1) = e^{e^{x_1}}$ and $g(x) = e^{e^{x_1+\sum_{j=2}^{N-1}\cos^j(x_j)}}$ do belong to $\mathsf{G}(\mathbb{R}^{N-1})$. Since this argument can be iterated, we see that the class $\mathsf{G}(\mathbb{R}^{N-1})$ contains smooth functions, bounded from below, that are neither uniformly continuous nor coercive, and with arbitrary large growth at infinity. Also note that, g_1 being convex, its epigraph satisfies a uniform exterior cone condition.

(3b) Assume $N \geq 2$ and let $g \in C^2(\mathbb{R}^{N-1})$ be any positive function such that $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$, then $\sqrt{g}$ is globally Lipschitz-continuous on $\mathbb{R}^{N-1}$ (see Lemma *I* in [Gla63] for $N = 2$). Therefore, we have $g \in \mathsf{G}(\mathbb{R}^{N-1})$. More generally, any $g \in C^2(\mathbb{R}^{N-1})$, bounded from below and such that $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$ is a member of the class $\mathsf{G}(\mathbb{R}^{N-1})$.
For instance, the function $g = g(x_1, \dots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1} \sin(jx_j)$ belongs to $\mathsf{G}(\mathbb{R}^{N-1})$ for any $N \geq 3$.

4. For $N = 2$, any continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\ell_- := \lim_{x \mapsto -\infty} g(x) \in (-\infty, +\infty]$ and $\ell_+ := \lim_{x \mapsto +\infty} g(x) \in (-\infty, +\infty]$ belongs to $\mathcal{G}$. Therefore, any quasiconvex (resp. quasiconcave) continuous function bounded from below belongs to $\mathcal{G}$. In particular, any monotone continuous function bounded from below and any convex functions bounded from below belongs to $\mathcal{G}$.

5. Assume $2 \leq n < N$ and let $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be a member of $\mathcal{G}$. Then, the function $\tilde{g} : \mathbb{R}^{N-1} \to \mathbb{R}$ defined by $\tilde{g}(x_1, \dots, x_{N-1}) = g(x_1, \dots, x_{n-1})$ satisfies the compactness property $(\mathcal{P})$, as a function on $\mathbb{R}^{N-1}$. Therefore, $\tilde{g} \in \mathcal{G}$, as a function on $\mathbb{R}^{N-1}$.
In particular, for $N \geq 2$, the functions $g(x_1, \dots, x_N) = e^{e^{x_1}}$ and $g(x_1, \dots, x_N) = e^{x_1} - 4\arctan(x_1) - 2$ belong to $\mathcal{G}$, are bounded from below and their epigraphs satisfy a uniform exterior cone condition.

6. Assume $N \geq 2$. Let $g \in \mathcal{G}$ and let $T : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ be a transformation of the form $T(x) = Ax + b$, where A an invertible real matrix and $b \in \mathbb{R}^{N-1}$. Then, $g \circ T \in \mathcal{G}$. In particular, this results applies when T is an isometry of $\mathbb{R}^{N-1}$.

7. Assume $N \geq 2$ and $\lambda \geq 0$. Let $g, \tilde{g} \in \mathcal{G}$ be bounded from below. Then, $\lambda g \in \mathcal{G}$ and $g + \tilde{g} \in \mathcal{G}$.
In particular, for $N \geq 2$, the function $g(x_1, \dots, x_N) = x_1^4 + e^{x_2}$ belongs to $\mathcal{G}$, is bounded from below and its epigraph satisfies a uniform exterior cone condition.

8. By combining items (1)-(7), one can easily build further examples of functions g belonging to $\mathcal{G}$ in any dimension $N \geq 2$. In particular, one can construct members of $\mathcal{G}$ bounded from below, that are neither uniformly continuous nor coercive, with arbitrary large growth at infinity, and such that their epigraphs satisfy a uniform exterior cone condition.

In view of the above discussions, we can now state the monotonicity results for continuous epigraphs defined by functions g belonging to the class $\mathcal{G}$. Let us start with the extension of Theorem 0.1.6 and Corollary 0.1.1.

Theorem 0.1.15 (B-Farina-Sciunzi). *Let $N \geq 2$ and let Ω be an epigraph bounded from below and defined by a function $g \in \mathcal{G}$. Assume $f \in Lip_{loc}([0, +\infty))$ with*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0. \tag{0.1.5}$$

(i) If $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ is a distributional solution to (SPE) *which is bounded and uniformly continuous on finite strips.*
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.
If f is globally Lipschitz-continuous, the same conclusion holds true only under the sole assumption that u is uniformly continuous on finite strips.

(ii) If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical solution of (SPE) *such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

If we also assume that the epigraph satisfies a uniform exterior cone condition, then we can prove the following extensions of Theorem 0.1.7 and Theorem 0.1.8.

Theorem 0.1.16 (B.-Farina-Sciunzi). *Let $N \geq 2$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.*

(i) Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE) *with at most exponential growth on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

(ii) Assume $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE) *which is bounded on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Note that Theorems 0.1.15 and 0.1.16 apply to the explicite examples of functions g provided in the above items (1)-(7). They also recover the results in the very recent preprint [GMS24], where the monotonicity is proved for some classical solutions to (SPE) with $f(t) = t^q$ and $q \geq 1$.[9]

We would like to point out that in all the results we have described before, we have always assumed the Lipschitz character of f (local or global), except for Theorem 0.1.14, this assumption is sharp, as illustrated by the following example:

Example 0.1.1. Assume $N \geq 2$ and let Ω be the half-space $\{x \in \mathbb{R}^N : x_N > 0\}$.
The bounded function

$$u(x) = \begin{cases} 1 - (x_N - 1)^4 & \text{if } \; 0 \leq x_N \leq 1, \\ \quad 1 & \text{if } \; x_N > 1, \end{cases} \tag{0.1.6}$$

[9]It is immediate to see that any semicoercive and continuous function is bounded from below and belongs to our class $\mathcal{G}$ (see items (5) and (6) above). This observation and Lemma A.2 in [GMS24] also show that the convex epigraph case is covered by the techniques we have developed in section 0.1

is a classical C^2 solution to (SPE), where f is given by the following non-increasing function

$$f(t) = \begin{cases} 12 & \text{if} \quad t < 0, \\ 12\sqrt{1-t} & \text{if} \quad 0 \leq t \leq 1, \\ 0 & \text{if} \quad t > 1. \end{cases} \tag{0.1.7}$$

Note that f is globally Hölder-continuous, but not locally Lipschitz-continuous on $\mathbb{R}$ and that $\frac{\partial u}{\partial x_N} = 0$ in $\{(x', x_N) \in \mathbb{R}^N,\, x_N > 1\}$.

Furthermore, since f is non-increasing, this example proves that the first conclusion of Theorem 0.1.14, namely that u is non-decreasing, is sharp.

0.1.4 Some applications to classification and non-existence results

In this subsection, we apply our monotonicity results to prove some classification and nonexistence results for the problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{SPE+}$$

where, Ω is an epigraph bounded from below.

In this context, the first result that we quote is due to M.J. Esteban and P.L. Lions. Indeed, in [EL82], they prove the following:

Theorem 0.1.17 (Esteban-Lions). *Let Ω be a smooth enough coercive epigraph and $f \in Lip_{loc}([0, +\infty))$ such that $f(0) \geq 0$. Assume that there exists a solution $u \in C^2(\overline{\Omega})$ to* (SPE+) *satisfying*

$$\lim_{\substack{x\in\Omega, \\ |x|\longrightarrow\infty}} u(x) = 0.$$

Then $u \equiv 0$.

Later on, A. Farina, in [Far07], provide a classification for solutions to the Lane-Emden equation $-\Delta u = u^p$ in a smooth coercive epigraph in any dimension.

Theorem 0.1.18 (Farina). *Let $0 < \alpha < 1$ and let Ω be a $C^{2,\alpha}$ coercive epigraph. Let $u \in C^2(\overline{\Omega})$ be a solution to*

$$\begin{cases} -\Delta u = u^p & \textit{in} \quad \Omega, \\ u \geq 0 & \textit{in} \quad \Omega, \\ u = 0 & \textit{on} \quad \partial\Omega. \end{cases} \tag{0.1.8}$$

with

$$\begin{cases} 1 < p < +\infty & \textit{if } N \leq 10, \\ 1 < p < p_{JL}(N) := \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \textit{if } N \geq 11. \end{cases}$$

Then,

$$u \equiv 0 \quad \textit{in } \Omega.$$

If one further assumes that u is a **`bounded`** *solution to* (0.1.8) *and*

$$\begin{cases} 1 < p < +\infty & \text{if } N \leq 11, \\ 1 < p < p_{JL}(N-1) & \text{if } N \geq 12. \end{cases}$$

Then,

$$u \equiv 0.$$

This Theorem is the first classification result of possibly unbounded solutions to (0.1.8) in a smooth epigraph.

In [DF22], L. Dupaigne and A. Farina consider bounded solutions of (SPE+) in locally Lipschitz-continuous coercive epigraph where f is a positive function of class C^1 for $2 \leq N \leq 11$. In this framework, they prove the following:

Theorem 0.1.19 (Dupaigne-Farina). *Let $\Omega \subset \mathbb{R}^N$ denote a locally Lipschitz-continuous coercive epigraph and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a bounded solution of* (SPE+).
Assume that $f \in C^1([0,+\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$. Then $f(0) = 0$ and $u \equiv 0$

Differently from Theorem 0.1.17 of M.J Esteban and P.L Lions above, in Theorem 0.1.19, there is no assumption on the smoothness of Ω. Nevertheless, Theorem 0.1.19 applies to C^1 nonlinearity f and in dimensions $2 \leq N \leq 11$.

Theorem 0.1.19 applies to $f(t) = t^p$, hence it recovers the second conclusion of Theorem 0.1.18 for $N \leq 11$.

To prove this Theorem, the authors use the boundedness and the monotonicity of solutions to (SPE+) (which comes from Theorem 0.1.12) in order to ensures that the function

$$v(x_1, \cdots x_{N-1}) = \lim_{x_N \to +\infty} u(x', x_N)$$

is a classical stable[10] solution to $-\Delta v = f(v)$ in $\mathbb{R}^{N-1}$. The conclusion of Theorem 0.1.19 follows from a Liouville-type result valid in the whole space up to dimension 10 (see Theorem 1 in [DF22]).

In our first article (see chapter 1), thanks to our monotonicity results established for epigraphs defined by a function $g \in \mathcal{G}$ (see 0.5.1 in subsection 0.1.3), we are able to use the same idea to prove the following result:

Theorem 0.1.20 (B.-Farina-Sciunzi). *Let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to* (SPE+) *Assume that $f \in C^1([0,+\infty))$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$, then $u \equiv 0$ and $f(0) = 0$.*

[10]that is, for every $\phi \in C^1_c(\mathbb{R}^{N-1})$, we have

$$\int_{\mathbb{R}^{N-1}} f'(v)\phi^2 \leq \int_{\mathbb{R}^{N-1}} |\nabla \phi|^2.$$

The previous theorem remains true even for $N \geq 12$, if we add an assumption about the behaviour of f at the origin. In this case, the desired conclusion is obtained by making use of some Liouville-type theorems for stable solutions established in [Far07], [DF10] and [Far15]. The desired results are the content of Theorem 0.1.21 and Theorem 0.1.22 below.

Theorem 0.1.21 (B.-Farina-Sciunzi). *Assume $N \geq 12$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.*
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to (SPE+) *where $f \in C^1([0,+\infty))$ satisfies $f(t) > 0$ for $t > 0$ and $\liminf_{t\to 0^+} \frac{f(t)}{t^s} > 0$, for some $s \in \left[0, \frac{N-3}{N-5}\right)$.*
Then $u \equiv 0$ and $f(0) = 0$.

Notice that Theorem 0.1.20 and Theorem 0.1.21 immediately imply the following non-existence result when $f(0) > 0$.

Corollary 0.1.4 (B.-Farina-Sciunzi). *Assume $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition. If $f \in C^1([0,+\infty))$ satisfies $f(t) > 0$ for $t \geq 0$, then problem* (SPE+) *does not admit any bounded distributional solution of class $C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$.*

The next result concerns the following natural class of nonlinearities introduced in [DF10]:

$$\begin{cases} f \in C^1([0,+\infty)) \cap C^2((0,+\infty)), \quad f(0) = 0, \\ \\ f > 0, \ \textit{nondecreasing and convex in } (0,+\infty) \\ \\ \qquad s.t. \qquad \lim_{u\to 0^+} \frac{f'(u)^2}{f(u)f''(u)} := q_0 \in [0,+\infty] \end{cases} \tag{0.1.9}$$

where, in the latter, we agree to set $\frac{f'(u)^2}{f(u)f''(u)} = +\infty$ if $f''(u) = 0$.
Notice that, we necessarily have $q_0 \in [1,+\infty]$ (see Lemma 1.4 in [DF10]).

The typical representative of this class is given by the function $f(u) = u^p$, $p > 1$. In this case $\frac{f'(u)^2}{f(u)f''(u)} = \frac{p}{p-1}$ and so q_0 coincides with the conjugate exponent of p. Consequently, if we define $p_0 \in [1,+\infty]$ as the conjugate exponent of q_0 by $\frac{1}{p_0} + \frac{1}{q_0} = 1$, we have that the exponent p_0 can be considered as a "measure" of the flatness of f at the origin.
Other members of the preceding class are provided by the functions $f_n(u) = e^u - \sum_{k=0}^n \frac{u^k}{k!}$, where $n \geq 1$ is an integer. It is easily seen that $q_0(f_n) = \frac{n+1}{n}$ and so $p_0(f_n) = n + 1$.

Theorem 0.1.22 (B.-Farina-Sciunzi). *Assume $N \geq 12$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.*
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to (SPE+) *where f satisfies* (0.1.9)*.*
Suppose that p_0, the conjugate exponent of q_0, satisfies

$$1 \leq p_0 < p_{JL}(N-1), \tag{0.1.10}$$

where p_{JL} is the Joseph-Lundgren stability exponent given by

$$p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

Then $u \equiv 0$.

Note that the preceding theorem applies to $f(u) = u^p$ if $1 \leq p < p_{JL}(N-1)$ and to f_n if $n + 1 < p_{JL}(N-1)$.

Next we state the following immediate consequence of Theorem 0.1.13.

Corollary 0.1.5 (**B.-Farina-Sciunzi**). *When the epigraph Ω is coercive, the conclusion of Theorems 0.1.20-0.1.22 and Corollary 0.1.4 holds true under the sole assumption of continuity of g, i.e., we do not need to require the uniform exterior cone condition for the epigraph.*

Corollary 0.1.5 recovers and completes Theorem 0.1.18.

We conclude this subsection with the following classification result for solutions of (SPE+) that tend to zero at infinity.

Theorem 0.1.23 (**B.-Farina-Sciunzi**). *Assume $N \geq 2$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.*
Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (SPE+) *such that*

$$\lim_{\substack{x \in \Omega, \\ |x| \longrightarrow \infty}} u(x) = 0. \tag{0.1.11}$$

Then $u \equiv 0$ and $f(0) = 0$.
When the epigraph Ω is coercive, the above conclusion holds true under the sole assumption of continuity of g.

Note that, in the previous result, we make no assumptions on f (beside $f \in Lip_{loc}([0, +\infty))$) or the boundedness of u.

The above result recovers and improves Theorem 0.1.17, where the conclusion has been obtained for a smooth coercive epigraph.

0.2 New comparison principles in unbounded sets

Our new monotonicity results described in the previous section are based on the moving plane method suitably adapted to the epigraph geometry. Let us explain this in more detail.

If we consider a solution u to problem (SPE), the moving plane method consists in proving the following equality:

$$\Lambda := \{t > 0 \ : \ u \leqslant u_\theta \ \text{ in } \ \Sigma_\theta\,, \ \forall\, 0 < \theta < t\} = (0, +\infty), \tag{0.2.1}$$

where $\Sigma_\theta = \{x = (x', x_N) \in \Omega,\ x_N < \theta\}$ and $u_\theta(x) = u(x', 2\theta - x_N)$.

This means that, for any cap Σ_t, $t > 0$, the solution u lies below u_t, its reflection with respect to the hyperplane of equation $\{x_N = t\}$. It is easy to see that if (0.2.1) is true, then u is non-decreasing in the x_N- direction. Finally, the strict monotonicity of u follows from the application of Hopf's lemma.

In order to show (0.2.1), we first have to prove that Λ is not empty. When Σ_θ is bounded, this is a consequence of a suitable application of the classical maximum principle. In our situation, the cap Σ_θ might be unbounded (possibly disconnected and of infinite measure) and u is not supposed to be bounded. To overcome these additional difficulties, in our first article (see chapter 1), we have proved some new comparison principles for a large class of unbounded Euclidean subsets, that we call *sets with good sections* (see subsection 0.2.1 below). In particular, these results generalize those already existing in the special case of finite strips, which we recall below.

In [Far20], A. Farina performs a new comparison principle for functions defined in strips of the form $\mathbb{R}^{N-1} \times [a, b]$. Indeed, he shows the following (see Theorem 2.1 in [Far20]):

Theorem 0.2.1 (**Farina**).

1) Let $N \geq 2$, $M > 0$ and assume that $f \in Lip_{loc}([0, +\infty))$. Then there exists $\varepsilon = \varepsilon(f, M) > 0$ such that, for any $(a, b) \subset \mathbb{R}$ with $0 < b - a < \varepsilon$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$ satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathbb{R}^{N-1} \times (a, b), \\ |u|, |v| \leq M & in \quad \mathbb{R}^{N-1} \times (a, b), \\ u \leq v & on \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

we have

$$u \leq v \quad in \quad \mathbb{R}^{N-1} \times (a, b).$$

2) Let $N \geq 2$ and assume that $f \in Lip([0, +\infty))$. Then there exists $\varepsilon = \varepsilon(f) > 0$ such that, for any $(a, b) \subset \mathbb{R}$ with $0 < b - a < \varepsilon$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$, with at most polynomial growth at infinity and satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathbb{R}^{N-1} \times (a, b) \\ u \leq v & on \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

we have

$$u \leq v \quad in \quad \mathbb{R}^{N-1} \times (a, b).$$

3) Let $N \geq 2$ and assume that $f \in C^0([0, +\infty))$ is a non-increasing function. Then, for any $(a, b) \subset \mathbb{R}$ and any $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$, with at most polynomial growth at infinity and satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathbb{R}^{N-1} \times (a, b) \\ u \leq v & on \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

we have

$$u \leq v \quad in \quad \mathbb{R}^{N-1} \times (a, b).$$

In this context, H. Berestycki, L.A. Caffarelli and L. Nirenberg, in [BCN96], describe a maximum principle for `linear operator` and for functions with exponential growth defined in strips:

Theorem 0.2.2 (Berestycki-Caffarelli-Nirenberg). *In $\Omega = \mathbb{R}^{N-1} \times (a, b)$, suppose $w \in W^{2,\alpha}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ is a function satisfying*

$$\begin{cases} -\Delta w + c(x,y)w \leq 0 & in \;\; \Omega, \quad with\ |c| \leq \gamma, \\ \quad\quad w \leq 0 & in \;\; \partial\Omega, \end{cases}$$

and

$$w \leq Ce^{\mu|x|},$$

for some positive constants γ, C, and μ. There is a constant δ, depending only on N, γ and μ, such that if

$$0 < b - a < \delta,$$

then

$$w \leq 0 \ in \ \Omega.$$

0.2.1 Sets with good sections

In this sub-section, we present the sets on which we establish our new comparison principles.

Definition 0.2.1 (Sets with "good sections" in one direction). Assume $N \geq 2$. Let Ω be an open subset of $\mathbb{R}^N$ and let ν be a unit vector of $\mathbb{R}^N$.
We denote by $\mathbb{R}\nu$ the vector space spanned by the unit vector ν and by $\{\nu\}^\perp$ the orthogonal complement of ν in $\mathbb{R}^N$.

(i) We shall say that Ω is `locally bounded in the direction` ν if

$$\forall R > 0 \qquad C_\nu(R) = (B'(0', R) \times \mathbb{R}\nu) \cap \Omega \quad \text{is a bounded subset of } \mathbb{R}^N. \tag{0.2.2}$$

Here $B'(0', R)$ denotes the $N-1$-dimensional open ball of radius R centered at the origin $0'$ of $\{\nu\}^\perp$.

(ii) For every $x' \in \{\nu\}^\perp$ let us define the set $S^\nu_{x'} := (\{x'\} \times \mathbb{R}\nu) \cap \Omega$.

We shall say that Ω has `bounded section in the direction` ν if

$$\mathtt{S}_\nu(\Omega) := \sup_{x' \in \{\nu\}^\perp} \mathcal{L}^1(S^\nu_{x'}) < +\infty, \tag{0.2.3}$$

where $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure.

The positive number $\mathtt{S}_\nu(\Omega)$ will be called the `section of` Ω `in the direction` ν.

(iii) We shall say that Ω has `good section in the direction` ν if it is locally bounded in the direction ν and if it also has bounded section in the direction ν (that is, if it satisfies both (1.2.1) and (1.2.2)).

The class of sets defined above is very broad. Indeed, any open set contained in a finite strip clearly has good section for a suitable direction, but we can also find more sophisticated open sets not contained in strips like $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \,:\, |x| - h(x) < y < |x| + h(x)\} \cup \{(x,y) \in \mathbb{R}^2 \,:\, -|x| - h(x) < y < -|x| + h(x)\} \cup \{(x,y) \in \mathbb{R}^2, |y| < 1\}$, where $h(x) = sinh^{-1}(e^{-|x|})$, (see Figure 4). This set is not contains in any affine half-space nor any strips, besides it has infinite Lebesgue measure.

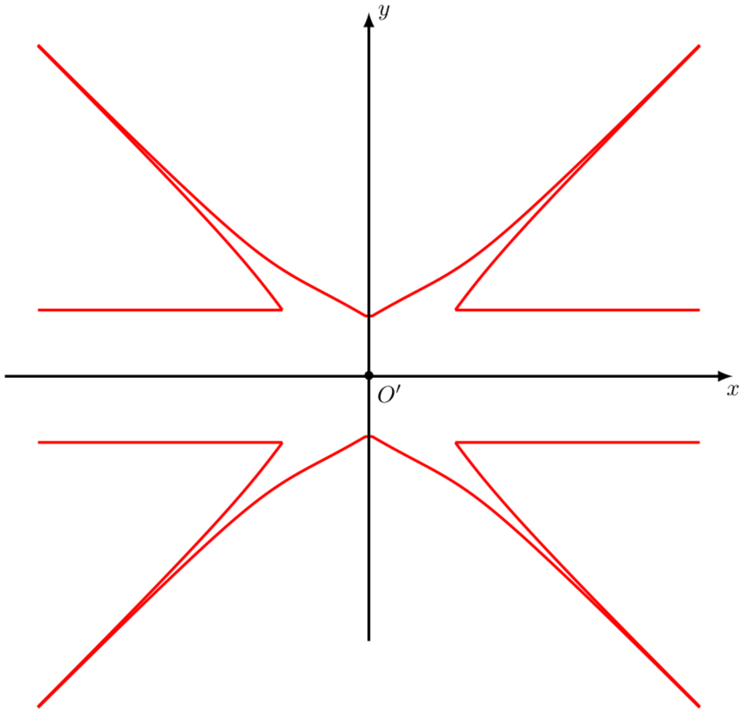

Figure 4: Ω_1

We also note that conditions (i) and (ii) above are not equivalent. Indeed, if we consider $\Omega_2 = \bigcup_{n\geq 1} \left\{(x,y) \in \mathbb{R}^2 \,:\, n < y < n + \frac{1}{2^n}\right\}$ and $\Omega_3 = \{(x,y) \in \mathbb{R}^2 \,:\, 0 < y < x^2\}$, then we can prove that Ω_2 has a bounded section in the direction e_2 ($\mathtt{S}_{e_2}(\Omega_2) = 1$) but is not bounded in direction e_2. Conversely, the set Ω_3 is locally bounded in the direction e_2 but its section in never bounded since $S_x^{e_2} = x^2$.

0.2.2 New comparison principles

We are now ready to introduce our new comparison principles:

Theorem 0.2.3 (B.-Farina-Sciunzi). *Assume $N \geq 2$. Let Ω be an open subset of $\mathbb{R}^N$ and let ν be a unit vector of $\mathbb{R}^N$.*

(i) Assume that Ω has good section in the direction ν. Let $f \in Lip_{loc}(\mathbb{R})$, $M > 0$ and

$u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *satisfying*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases} \tag{0.2.4}$$

Then, there exists $\varepsilon = \varepsilon(f, M) > 0$ *such that*

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \ in \ \Omega. \tag{0.2.5}$$

(ii) Assume that Ω *has good section in the direction* ν*. Let* $f \in C^0(\mathbb{R})$ *be a non-increasing function,* $M > 0$ *and* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *satisfying*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases} \tag{0.2.6}$$

Then, $u \leq v$ *in* Ω*.*

(iii) Assume that Ω *has bounded section in the direction* ν*. Let* $f \in Lip_{loc}(\mathbb{R})$*,* $M > 0$ *and* $u, v \in Lip_{loc}(\overline{\Omega})$ *satisfying*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in & \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in & \Omega, \\ |\nabla u|, |\nabla v| \leq M & a.e. \ in & \Omega, \\ u \leq v & on & \partial\Omega. \end{cases} \tag{0.2.7}$$

Then, there exists $\varepsilon = \varepsilon(f, M) > 0$ *such that*

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \ in \ \Omega. \tag{0.2.8}$$

More generally, we have the following results:

Theorem 0.2.4 (B.-Farina-Siunzi). *Assume* $\gamma \geq 0$*,* $\delta \geq 0$*,* $N \geq 2$ *and let* Ω *be an open subset of* $\mathbb{R}^N$ *with good section in the direction* e_N*, the last vector of the canonical base of* $\mathbb{R}^N$*, such that*

$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma |x_N|} dx_N \right) < +\infty. \tag{0.2.9}$$

Let $f = f_1 + f_2$*, with* $f_1 \in Lip(\mathbb{R})$ *and* $f_2 : \mathbb{R} \mapsto \mathbb{R}$ *be a non-increasing function.*

Let $a > 0$ *and* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *such that*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^\delta e^{\gamma |x|} & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases}$$

Then, there exists $\varepsilon = \varepsilon(L_{f_1}, \gamma) > 0$ *such that*

$$\mathtt{S}_{e_N}(\Omega) < \varepsilon \implies u \leq v \ in \ \Omega.$$

Here, L_{f_1} *denotes the Lipschitz constant of* f_1*.*

The following remarks are in order:

(i) These Theorems recover and complete Theorems 0.2.1 and 0.2.2.

(ii) Any open set Ω included in a strip $\{x = (x', x_N) \in \mathbb{R}^N \, : \, \alpha < x_N < \beta\}$, with $\alpha, \beta \in \mathbb{R}$, clearly satisfies the assumption (0.2.9) for every $\gamma, \delta \geq 0$.

(iii) Notice that the preceding theorem applies to the set Ω_1 (see figure (4)). Indeed, Ω_1 satisfies the assumption (0.2.9) for any $\delta \geq 0$ and any $\gamma \in [0, 1/2]$. .

The smallness assumption on the section of Ω is necessary for the validity of both Theorem 0.2.3 (item (i) and item (iii)) and Theorem 0.2.4. Indeed, the functions $u = \sin(y)$ and $v \equiv 0$ satisfy $-\Delta u - u = 0 = -\Delta v - v$ on the two-dimensional strip $\Omega = \{(x, y) \in \mathbb{R}^2 \, : \, 0 < y < \pi\}$ and $u \leq v$ on $\partial\Omega$, but the conclusion of the comparison principles fails.

When f is non-increasing on $\mathbb{R}$, the comparison principle holds even without the smallness assumption on the section of Ω. More precisely we have the following

Theorem 0.2.5 (B.-Farina-Sciunzi). *Assume $N \geq 2$ and let Ω be an open subset of $\mathbb{R}^N$ bounded in the direction e_N, the last vector of the canonical base of $\mathbb{R}^N$. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a non-increasing function and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ such that*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^{\delta} e^{\gamma|x|} & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega, \end{cases}$$

for some $a > 0$, $\delta \geq 0$ and $\gamma \in \left[0, \frac{\pi}{4\mathtt{s}_{e_N}(\Omega)\sqrt{e-1}}\right)$.
Then, $u \leq v$ in Ω.

That the growth assumption on u in Theorem 0.2.5 is necessary to ensure the comparison principle is seen from the following classical example involving harmonic functions on finite strips of $\mathbb{R}^2$, namely, when $f \equiv 0$. It is well-known that, for any integer $k \geq 1$, the function $u_k(x, y) = \sum_{m=1}^{k} (e^{mx} + e^{-mx}) \sin(mx)$ is harmonic on the strip $\Omega = \{(x, y) \in \mathbb{R}^2 \, : \, 0 < y < \pi\}$ and vanishes on $\partial\Omega$. Therefore, a restriction on the exponential growth must be prescribed in order to get the desired results.

The proofs of these results is inspired from the proof of Theorem 0.2.1 and rely on the choice of good test functions and on a delicate application of a Lemma from [BFP24].

0.2.3 Applications of our new comparison principles

The role of our new comparison principles is twofold:

1) they are crucial for proving our new monotonicity results on epigraphs. They implies that the moving plane method can be started regardless of the value of $f(0)$ and for a large class of unbounded domains. Specifically we have the following:

Corollary 0.2.1 (Starting the moving planes method). *Assume $N \geq 2$. Let Ω be a domain contained in the upper half-space $\mathbb{R}^N_+$ and also assume that Ω satisfies property $\mathcal{R}$, that we recall below:*

$\mathcal{R}$: Ω ***contains reflection (with respect to the hyperplane*** $\{x_N = \theta\}$***) of all strips*** $\Omega \cap \{x_N < \theta\}$***.***

Let u be a distributional (resp. classical) solution to (SPE)*. Suppose that one of the following assumptions is in force :*

(i) $f \in Lip_{loc}([0, +\infty))$ and u is bounded on $\Omega \cap \{x_N < R\}$;

(ii) $f \in Lip([0, +\infty))$ (resp. a non-increasing function) and u has at most exponential growth on $\Omega \cap \{x_N < R\}$.

Then there exists $R_0 \in (0, R)$ such that $\Omega \cap \{x_N < R_0\} \not\equiv \emptyset$ and

$$\frac{\partial u}{\partial x_N} > 0 \quad in \quad \Omega \cap \{x_N < R_0\}\,.$$

The above result extends the one proved in [Far20] for the special case of a half-space.

For sake of clarity (and simplicity) we have stated the above result in the case $\Omega \subseteq \mathbb{R}^N_+$. Since the considered problem is invariant by isometry, it is clear that the same result holds if Ω is contained in an affine open half-space H. In this case, the monotonicity will be obtained with respect to ν, the inner normal to H.

2) They allow us to prove some new results of uniqueness and symmetry for solutions (possibly unbounded and sign- changing) to the homogeneous Dirichlet BVP for the semilinear Poisson equation in fairly general unbounded domains.

Corollary 0.2.2 (B.-Farina-Sciunzi). *Let Ω and f be as in the statement of Theorem 0.2.3 (resp. Theorem 0.2.4 or Theorem 0.2.5). Then, the Dirichlet problem*

$$\begin{cases} -\Delta u = f(u) & in\ \mathcal{D}'(\Omega), \\ \quad u = 0 & on\ \partial\Omega, \end{cases} \tag{0.2.10}$$

has, at most, one solution u satisfying the regularity and the growth conditions stated in Theorem 0.2.3 (resp. Theorem 0.2.4 or Theorem 0.2.5).
In particular, if $f(0) = 0$, the function $u \equiv 0$ is the only solution to the Dirichlet problem (0.2.10)*.*

The examples considered after Theorem 0.2.5, prove that the preceding result is no longer true if either the assumptions on the size of Ω or those on the growth of u are not met.

An immediate consequence of Corollary 0.2.2 is the following general symmetry result for solutions to the homogeneous Dirichlet problem (0.2.10).

Corollary 0.2.3 (B.-Farina-Sciunzi). *Let Ω, f be as in the statement of Corollary 0.2.2 and assume that the Dirichlet problem*

$$\begin{cases} -\Delta u = f(u) & in\ \mathcal{D}'(\Omega), \\ \quad u = 0 & on\ \partial\Omega, \end{cases} \tag{0.2.11}$$

has a solution u satisfying the regularity and the growth conditions stated in Corollary 0.2.2.

Let ρ be an isometry of $\mathbb{R}^N$, $N \geq 2$. If Ω is invariant with respect to ρ, i.e., Ω satisfies $\rho(\Omega) = \Omega$, then the solution u inherits the same symmetry, namely, $u(x) = u(\rho(x))$ for any $x \in \Omega$.

Let us illustrate the above result on the *torsion problem* for an infinite solid bar. Let us start with the case of an infinite solid straight bar with spherical cross section. That is, the study of the solutions to the problem

$$\begin{cases} -\Delta u = K & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2.12}$$

where $\Omega = \Omega_{K,R} := \mathbb{R}^{N-K} \times B_R^K$, where $1 \leq K < N$ is an integer, and B_R^K denotes the open ball of $\mathbb{R}^K$ centered at the origin and of radius $R > 0$.

Since $\Omega_{K,R}$ is invariant with respect to any translation in the variables $x_1, \dots, x_{N-K}$ and also to any rotation in the variables $x_{N-K+1}, \dots, x_N$, Corollary 0.2.3 (applied with the non-increasing function $f \equiv K$.) tell us that the unique solution u of (0.2.12), with subexponential growth at infinity, must be independent of $x_1, \dots, x_{N-K}$ and radially symmetric in the variables $x_{N-K+1}, \dots, x_N$. It is therefore easy to check that the function

$$u(x) = \frac{R^2 - (x_{N-K+1}^2 + \ldots + x_N^2)}{2}, \qquad x \in \Omega_{K,R}$$

is the only solution to the problem (0.2.12) with subexponential growth at infinity.

The previous analysis extends to general infinite solid bars (not necessarily straight). For instance, it applies to the following examples :

1) $\Omega = \mathbb{R}^{N-K} \times \omega$, where $1 \leq K < N$ is an integer and ω is an open bounded subset of $\mathbb{R}^K$. In this case, the unique solution u of (0.2.12) with subexponential growth at infinity, depends only on the variables $x_{N-K+1}, \dots, x_N$.

2) $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -\varphi(x_1) < x_2 < \varphi(x_1)\}$, where $\varphi : \mathbb{R} \mapsto (0, +\infty)$ is a bounded continuous function. In this case the solution u must be symmetric with respect to the line $x_2 = 0$.

3) Ω is a domain of revolution of $\mathbb{R}^N$,$N \geq 3$, i.e., $\Omega = \left\{x \in \mathbb{R}^N : \sqrt{x_2^2 + \ldots + x_N^2} < \varphi(x_1)\right\}$, where $\varphi : \mathbb{R} \mapsto (0, +\infty)$ is a bounded continuous function. Here, u must have the form $u = u(x_1, \sqrt{x_2^2 + \ldots + x_N^2})$.

Also observe that, if φ is T-periodic, with $T > 0$, then the solution u is T-periodic in the x_1 variable.

It is clear that the above discussions and results also hold true for general non linear function f satisfying the assumptions of Corollary 0.2.3.

0.3 Flattening results for the Serrin's overdetermined problem (SOP) in epigraphs

Thanks to our new monotonicity results, we are able to provide new information concerning the shape of Ω and the behavior of solution to the Serrin's overdetermined problem (SOP), i.e. to the problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{on} \quad \partial\Omega. \end{cases} \tag{SOP}$$

In particular, in this section, we focus on Conjecture D in epigraphs. In the following subsections, we first present existing the works concerning this topic and then we introduce our new results.

0.3.1 Serrin's overdetermined problem with Allen-Cahn type non-linearity

In this subsection, we consider problem (SOP) where f is an Allen-Cahn type function (see Definition 0.1.1). In this framework, H. Berestycki, L.A. Caffarelli and L. Nirenberg, in their article [BCN97b] prove the following result:

Theorem 0.3.1 (Berestycki-Caffarelli-Nirenberg). *Let f be an Allen-Cahn type function (see (0.1.1)). Let $g : \mathbb{R}^{N-1} \to \mathbb{R}$ be a globally-Lipschitz continuous function of class C^2 satisfying the following condition:*

$$\textit{for any } \tau \in \mathbb{R}^{N-1} \qquad \lim_{x \in \mathbb{R}^{N-1}, |x| \to \infty} g(x+\tau) - g(x) = 0 \tag{0.3.1}$$

and let Ω be its epigraph. If there exists a bounded solution u to problem (SOP), then Ω is an upper half-space and u is symmetric and monotone.

This gives a partial answer to Conjecture D. Indeed, since problem (SOP) is invariant by euclidean isometries, all the natural solutions of (SOP) corresponding to the affine functions $g(x) = a \cdot x + b$, $a \neq 0$, are ruled out by the additional condition on g at infinity.

The theorem above holds in any dimension, nevertheless the proof is crucially based on the fact that f is an Allen-Cahn type function, as well as on the on the additional condition (0.3.1) on the epigraph at infinity.

In 2010, A. Farina and E. Valdinoci [FV10] introduced a new geometric tool that allowed them to prove Conjecture D without any additional condition on g, but under the dimensional constraint $N \leq 3$

Theorem 0.3.2 (Farina-Valdinoci). *Let $N = 2, 3$ and let Ω be an epigraph of $\mathbb{R}^N$ with C^3 and globally Lipschitz-continuous boundary. Let $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ be a solution of*

(SOP) *where f is an Allen-Cahn type function.*
Then, we have that $\Omega = \mathbb{R}^N_+$ up to isometry and that there exists $u_0 : (0, +\infty) \to (0, +\infty)$ in such a way that

$$u(x_1, \cdots, x_N) = u_0(x_N) \quad \text{for any } (x_1, \cdots, x_N) \in \mathbb{R}^N_+.$$

Subsequently, the new geometric tool introduced in [FV10] has been used to study Serrin's overdetermined problem on general domains of a Riemannian manifold (see [FMV13]).

In general, we cannot expect to prove Conjecture D in any dimension, since in [PPW15], M. Del Pino, F. Pacard and J. Wei build a smooth epigraph Ω (which is not an affine half-space) such that, if $N \geq 9$, the overdetermined problem (SOP) admits a smooth, bounded and monotone solution. Moreover, the authors of [PPW15] noticed that the epigraph they built is not globally Lipschitz-continuous.

In [WW19], K. Wang and J. Wei investigate Conjecture D, where f is a *specified Allen-Cahn-type non-linearity,* that is, when$f$ satisfies the following definition:

Definition 0.3.1. A function f is a *specified Allen-Cahn-type non-linearity* if the function

$$W(t) := \int_0^1 f(s)ds - \int_0^t f(s)ds, \qquad t \geq 0,$$

belongs in $C^2([0, +\infty))$ and satisfies the following conditions :

W1) $W \geq 0$, $W(1) = 0$ and $W > 0$ in $[0, 1)$;

W2) $W' < 0$ in $(0, 1)$, and either $W'(0) \neq 0$ or

$$W'(0) = 0 \quad \text{and} \quad W''(0) \neq 0;$$

W3) $\exists\, \delta_2 \in (0, 1), \exists\, \kappa > 0$ such that $W'' \geq \kappa$ in $[\delta_2, 1]$;

W4) $\exists\, p > 1, \exists\, c > 0$ such that $W'(t) \geq c(t-1)^p$ for $t > 1$.

The typical representative of the class defined above is provided by $f(t) = t - t^3$, the Allen-Cahn nonlinearity. In this case, we have that $W(t) = \frac{(1-t^2)^2}{4}$. We also note that, a specified Allen-Cahn-type non-linearity is a particular case of an Allen-Cahn type function (see (0.1.1)).

For this type of functions, in [WW19], K. Wang and J. Wei prove Conjecture D under various assumptions on the epigraph Ω. Specifically, they prove the following result:

Theorem 0.3.3 (**Wang-Wei**). *Assume $N \geq 2$ and let f be a specified Allen-Cahn type nonlinearity. Let g be a sufficiently smooth globally Lipschitz-continuous function and let Ω be its epigraph. If Serrin's overdetermined problem* (SOP) *has a solution, then Ω must be a half-space and u is symmetric.*

This theorem holds true in any dimension. It also confirms that the epigraph previously constructed by M. Del Pino, F. Pacard and J. Wei in dimension $N \geq 9$ cannot be globally Lipschitz-continuous.

The authors of [WW19] also prove Conjecture D in dimension $N \leq 8$, when Ω is a smooth epigraph, not necessarily globally Lipschitz-continuous, but imposing that the considered solution u is monotone.

Theorem 0.3.4 (Wang-Wei). *Let f be a specified Allen-Cahn type nonlinearity. Let Ω be a sufficiently smooth epigraph and let $u \in C^2(\overline{\Omega})$ be a solution of* (SOP) *such that,*

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in } \Omega.$$

If $N \leq 8$, then Ω is a half space and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ such that, after a translation, $u(x) = u_0(x \cdot e)$ for some unit vector e.

We are now ready to state our results. In our second article (see chapter 2), we also establish new flattening results for specified Allen-Cahn non-linearities. Specifically, we prove:

Theorem 0.3.5 (B.-Farina). *Assume $N \geq 2$ and let $f \in C^1([0, +\infty))$ be a specified Allen-Cahn type non-linearity. Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to* (SOP) *satisfying*

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega. \tag{0.3.2}$$

Let $\Omega \subset \mathbb{R}^N$ be an epigraph defined by a function $g \in C^1(\mathbb{R}^{N-1})$ and, if $N \geq 9$ let us also assume that g satisfies :

a) $\exists\, C > 0$ for which

$$g(x') \geq -C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1};$$

or

b) $\exists\, C > 0$ for which

$$g(x') \leq C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1}.$$

Then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+.$$

The following remarks are in order :

(i) In the above Theorem we do not assume that u is bounded. Indeed, this property is automatically satisfied by any solution to (SPE) on general domains, possibly unbounded, when f is a specified Allen-Cahn type non-linearity. See Corollary 2.5.1.

(ii) When $N \leq 8$, the above Theorem recovers Theorem 0.3.4 discussed above. We also notice that, when g is globally Lipschitz-continuous, the monotonicity assumption (0.3.2) is automatically satisfied in any dimension $N \geq 2$ thanks to item (i) and Theorem 0.1.5. Since the Lipschitz character of g implies the validity of the additional condition a) (actually also b)), we see that the above Theorem also recovers Theorem 0.3.3 . When g is coercive, the monotonicity assumption (0.3.2) is automatically satisfied in any dimension $N \geq 2$ thanks to Theorem 0.1.4 of M.J Esteban and P.L. Lions and, since the additional condition a) is clearly in force, we see that also Theorem 1.3 in [WW19] is a particular case of Theorem 0.3.4 above.

(iii) The additional assumptions for $N \geq 9$ have the following geometrical interpretation: either $\mathbb{R}^N \setminus \Omega$ contains a cone or Ω contains a cone (or equivalenty, either the graph of g lies above the graph of a cone or the graph of g lies below the graph of a cone).

(iv) The above remarks also show that the epigraph constructed by M. Del Pino, F. Pacard and J. Wei in [PPW15] does not contain a cone (and neither does its complementary) and therefore that this epigraph is neither globally Lipschitz-continuous (as already observed in [PPW15]), nor contained in an affine half-space.

If we further assume that Ω is a smooth enough epigraph bounded from below, then the monotonicity condition (0.3.2) is automatically satisfied, thanks to our Theorem 0.1.16. Hence, we have the following result:

Theorem 0.3.6 (B.-Farina). *Assume $N \geq 2$ and let Ω be an epigraph bounded from below and defined by a function $g \in \mathcal{G} \cap C^1(\mathbb{R}^{N-1})$.*
Let $f \in C^1([0, +\infty))$ be a specified Allen-Cahn type non-linearity and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (SOP).
Then, $\Omega = \mathbb{R}^N_+$ up to a vertical translation and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+.$$

Therefore, Theorem (0.3.6) proves Conjecture D for a broad family of continuous epigraphs bounded from below, not necessarily globally Lipschitz-continuous, when f is a specified Allen-Cahn type function. A natural question is whether the same result can be obtained for a more general function f.

0.3.2 Serrin's overdetermined problem with general non-linearity

In the mathematical literature, there are few results concerning problem (SOP) and Conjecture D, when f is a general locally Lipschitz-continuous function.

In this respect, in [RRS17], A. Ros, D. Ruiz and P. Sicbaldi solve Conjecture D in dimension $N = 2$ for smooth enough epigraphs, precisely they show the following result:

Theorem 0.3.7 (Ros-Ruiz-Sicbaldi). *Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class $C^{1,\alpha}$ and let Ω be its epigraph. Assume that there exists a bounded solution to problem* (SOP) *for* *some locally Lipschitz-continuous function* $f : [0, +\infty) \to \mathbb{R}$*. Then* Ω *is a half-plane and* u *depends only on one variable.*

The theorem describe in [RRS17] is more general, since it applies to domains Ω of class $C^{1,\alpha}$ such that $\partial\Omega$ is unbounded and connected.

In all the results presented above, as well as in Conjecture D, the solutions of the (SOP) problem are always assumed to be bounded (and this fact was important for carrying out the proofs.) A first rigidity result concerning unbounded solutions of the problem (SOP) was provided in [FV10]. The necessity to consider also unbounded solutions to Serrin's overdetermined problem (SOP) comes from the fact that, for some natural functions f, the only solutions to (SOP) are unbounded. This is well illustrated by the case of harmonic functions, that is when $f \equiv 0$. In this case, there might be no bounded solution to (SOP), whereas the function $u(x) = x_N$ is a positive unbounded harmonic function on the half-space $\mathbb{R}^N_+$, with zero Dirichlet boundary condition. In [FV10], authors establish a rigidity result for possibly unbounded solutions to (SOP) in dimension 2, indeed they show the following theorem:

Theorem 0.3.8 (Farina-Valdinoci). *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a function of class* C^3 *and* Ω *be its epigraph. Let* f *be a locally Lipschitz-continuous function and* $u \in C^2(\overline{\Omega})$ *be a solution of* (SOP)*, with* $|\nabla u| \in L^\infty(\Omega)$*. Assume that* u *satisfies the following condition:*

$$\frac{\partial u}{\partial x_2} > 0 \quad \text{in } \Omega.$$

Then, Ω *must be a half-plane, and there exist* $\omega \in \mathbb{S}^1$ *and* $u_0 : \mathbb{R} \to (0, +\infty)$ *in such a way that* ω *is normal to* $\partial\Omega$ *and* $u(x) = u_0(\omega \cdot x)$ *for any* $x \in \Omega$*.*

To prove their theorem, A. Farina and E. Valdinoci developed a new geometric approach for problem (SOP) that allow them to obtain a rigidity result for domains Ω which are connected and of class C^3 (not necessarily epigraphs).

We are now ready to introduce our contributions about problem (SOP) for a general function f. They have been obtained in our second article (see chapter 2), where we proved new flattening and symmetry results for possibly unbounded solutions of (SOP) in epigraphs bounded from below in dimensions $N \leq 3$.

The proofs of the following results are obtained by combining the geometric approach developed in [FV10] and our new monotonicity results described in the first section (see 0.1).

Let us begin with a rigidity result for bounded solutions to the overdetermined problem (SOP) in a globally Lipschitz-continuous epigraph bounded from below of $\mathbb{R}^3$:

Theorem 0.3.9 (B.-Farina). *Let* $\Omega \subset \mathbb{R}^3$ *be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class* C^3*.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *be a bounded solution to* (SOP)*, where* $f \in C^1([0, +\infty))$ *satisfies* $f(0) \geq 0$ *and*

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t), \tag{0.3.3}$$

here F denotes the primitive of f vanishing at 0*.*

Then, $\Omega = \mathbb{R}^3_+$ *up to a vertical translation and there exists* $u_0 : [0, +\infty) \to (0, +\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Note that the assumption (0.3.3) is natural and somehow necessary. Indeed, if Conjecture (D) is true, i.e., if Ω is an affine half-space, then the Unique Continuation Principle (see Theorem 1 in [FV13]) ensures that the solution u must be one-dimensional. Next, a standard analysis of the first integral gives that $F(\sup u) > F(t)$, for any $t \in [0, \sup u)$, thus confirming the validity of (0.3.3).

We observe that condition (0.3.3) is clearly satisfied when:

(a) $f(t) \geq 0$, for any $t \geq 0$,

(b) there exist $\zeta > 0$, such that $f(t) \geq 0$ for any $t \in [0, \zeta]$ and $f(t) \leq 0$ for any $t \in [\zeta, +\infty)$.

We also note that, for $N = 2$, the conclusion of Theorem 0.3.9 remains valid without condition (0.3.3), even for unbounded solutions to (SOP). Actually, a more general result is true. This is discussed in the following theorem.

Theorem 0.3.10 (B.-Farina). *Let* $\Omega \subset \mathbb{R}^2$ *be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class* C^3 *and let* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *be a solution to* (SOP)*. Assume that one of the following hypotheses holds true:*

$$f \in Lip_{loc}([0, +\infty)), f(0) \geq 0 \text{ and } \nabla u \in L^\infty(\Omega); \tag{H1}$$

$$\begin{cases} f \in Lip_{loc}([0, +\infty)), \quad f(0) \geq 0, \quad \exists\, \zeta > 0 : f(t) \leq 0 \quad in \quad [\zeta, +\infty), \\ \exists\, \alpha \in [0, 1) : \quad u(x) = O(|x|^\alpha) \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{H2}$$

$$\begin{cases} f \in Lip([0, +\infty)), \quad f(0) \geq 0, \quad \exists\, \zeta > 0 : f(t) \geq 0 \quad in \quad [\zeta, +\infty), \\ u(x) = o(\ln |x|), \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{H3}$$

$$\begin{cases} f \in Lip([0, +\infty)), \quad f(0) \geq 0, \quad f(t) \leq 0 \quad in \quad (0, +\infty), \\ u(x) = o(|x| \ln^{\frac{1}{2}} |x|), \qquad as \quad |x| \longrightarrow \infty. \end{cases} \tag{H4}$$

Then, $\Omega = \mathbb{R}^2_+$ *up to a vertical translation and there exists* $u_0 : [0, +\infty) \to (0, +\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

We also establish a rigidity result for possibly unbounded solutions of (SOP) in dimension 3.

Theorem 0.3.11 (**B.-Farina**). *Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3 and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to* (SOP)*. Assume that one of the following assumptions holds true:*

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(t) \geq 0 \quad \textit{for any} \quad t \geq 0, \\ \\ u(x) = o(\ln|x|), \qquad \textit{as} \quad |x| \longrightarrow \infty; \end{cases} \tag{A1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad \exists\, \zeta \geq 0 \,:\, f(t) \geq 0 \quad \textit{on} \quad [0,\zeta] \quad \textit{and} \quad f(t) \leq 0 \quad \textit{on} \quad [\zeta,+\infty), \\ \\ \exists\, \alpha \in [0,1) \,: \quad u(x) = O(|x|^\alpha) \qquad \textit{as} \quad |x| \longrightarrow \infty; \end{cases} \tag{A2}$$

Then, $\Omega = \mathbb{R}^3_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

To the best of our knowledge, this is the first rigidity result for possibly unbounded solutions to (SOP) in epigraphs included in $\mathbb{R}^3$ and with natural general assumptions on f.

Proofs of these Theorems are crucially based on the following result, which follows from the use of the Poincaré-type formula and a detailed analysis of the level sets of solutions (see [FV10])):

Theorem 0.3.12 (**B.-Farina**). *Assume $N = 2,3$ and let $\Omega \subset \mathbb{R}^N$ be a domain of class C^3. Let $f \in Lip_{loc}([0,+\infty))$ and let $u \in C^2(\overline{\Omega})$ be a solution to* (SOP) *such that*

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad \textit{as} \;\; R \longrightarrow \infty. \tag{0.3.4}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{0.3.5}$$

then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \quad \forall x \in \mathbb{R}^N_+.$$

Note that this theorem remains true in any dimension, however energy condition (0.3.4) is less natural (and more difficult to be satisfied) in dimensions higher than three.

Thanks to our new monotonicity result (see Theorem 0.1.1) established in a sufficiently smooth epigraph bounded from below, and with $\mathfrak{c} \neq 0$, Theorems 0.3.9, 0.3.10 and 0.3.11 are still true, even if $f(0) < 0$. More precisely we have:

Theorem 0.3.13 (**B.-Farina**). *Let $\Omega \subset \mathbb{R}^2$ be a globally Lipschitz-continuous epigraph bounded from below and defined by a function $g \in C^3(\mathbb{R}) \cap \mathbb{C}^{1,\gamma}(\mathbb{R})$.*

Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (SOP) *with $\mathfrak{c} \neq 0$ and assume that one of the following hypotheses holds true:*

$$f \in Lip_{loc}([0,+\infty)), f(0) < 0 \text{ and } \nabla u \in L^\infty(\Omega); \tag{H1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad f(0) < 0, \quad \exists\, \zeta > 0 : f(t) \leq 0 \quad in \quad [\zeta, +\infty), \\ \exists\, \alpha \in [0,1) : \quad u(x) = O(|x|^\alpha) \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{H2}$$

Then, $\Omega = \mathbb{R}^2_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

Theorem 0.3.14 (B.-Farina). *Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz-continuous epigraph bounded from below and defined by a function $g \in C^3(\mathbb{R}^2) \cap \mathbb{C}^{1,\gamma}(\mathbb{R}^2)$.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a bounded solution to (SOP) *with $\mathfrak{c} \neq 0$.*
Let $f \in C^1([0,+\infty))$ be such that $f(0) < 0$ and

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t).$$

Then, $\Omega = \mathbb{R}^3_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

0.4 Classification results for solutions to semilinear Poisson equation in the half-space

In this section, we present a classification of possibly unbounded solutions to the following problem:

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{SPE+}$$

where f is a locally Lipschitz-continuous function on $[0,+\infty)$.

As in the previous sections, we first present the existing works concerning this problem, and then, we introduce our new results on this topic. These results are contained in our third article (see chapter 3).

Studying and classifying solutions to problem (SPE+) is natural for various reasons:

(i) the half-space is the simplest unbounded domain with an unbounded boundary in which one can study the Dirichlet problem for the semilinear Poisson equation.

(ii) Problem (SPE+) arises as a limit problem, after a blow-up procedure near the boundary, either when one wants to derive *a priori* estimates for positive solutions to some subcritical semilinear uniformly elliptic BVPs on smooth bounded domains ([GS81a]), or when one studies uniqueness results for classical solutions to the singular perturbation problem

$$-\varepsilon^2\Delta v = f(v) \quad \text{in} \quad \Omega, \qquad v > 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega, \tag{0.4.1}$$

where Ω is a smooth and bounded domain, f is a suitable smooth function and ε is a small and positive parameter (see for instance [Ang85] and [Dan09]).

Notice that, when $f(0) < 0$, even if solutions v considered in (0.4.1) are positive, given the absence of the strong maximum principle, the solution u of (SPE+) obtained after the scaling procedure is "only" non-negative in $\mathbb{R}^N_+$.

This motivates the choice to consider the non-negative solutions of (SPE+) (and not only the positive ones).

Finally, the need to also consider unbounded solutions to (SPE+) is well illustrated by the case of harmonic functions, that is when $f \equiv 0$. In this case, it is well-known that all the solutions to (SPE+) are given by $u(x) = \alpha x_N$, $\alpha \geq 0$.

The first result concerning the complete classification of the solutions to (SPE+) with $f \not\equiv 0$ was obtained by Gidas and Spruck in 1981. It concerns the function $f(u) = u^p$ and it was motivated by the the first situation described in item (ii) above.

Theorem 0.4.1 (Gidas-Spruck). *Assume $N \geq 2$ and let p be any real number satisfying $1 < p \leq \frac{N+2}{N-2}$. The only classical solution to*

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

is $u \equiv 0$.

The proof in [GS81a] makes use of the moving spheres method. In 1993, Berestycki, Caffarelli and Nirenberg ([BCN93]) observed that the proof in [GS81a] actually yields the following result

Theorem 0.4.2 (Berestycki-Caffarelli-Nirenberg). *Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (SPE+) *where $f \in Lip_{loc}([0, +\infty))$. If any of the two following conditions holds:*

(i) $N = 2$ and $f \geq 0$,

(ii) $N \geq 3$ and f satisfies

$$t \mapsto \frac{f(t)}{t^{(N+2)/(N-2)}} \quad \text{is nonincreasing on } (0, +\infty). \tag{0.4.2}$$

Then, either $u \equiv 0$, or there exists a function $u_0 : [0, +\infty) \to [0, +\infty)$ such that

$$u(x) = u_0(x_N) \qquad \text{for any } x \in \mathbb{R}^N_+ \tag{0.4.3}$$

and $u_0 > 0$, $u_0' > 0$ *on* $(0, +\infty)$.

Solutions of the form (0.4.3), that is, solutions depending only on the variable x_N, are called *one-dimensional* since they inherit the symmetry of the underlying domain $\mathbb{R}^N_+$. Clearly, the profile u_0 solves the one-dimensional problem

$$\begin{cases} -u_0'' = f(u_0) & \text{in} \quad (0, +\infty), \\ u_0(0) = 0. \end{cases} \tag{0.4.4}$$

Note that we necessarily have $f(0) \geq 0$ in the above Theorem 0.4.2. Recently, the following results about the one-dimensional symmetry of the solutions to (SPE+) have been obtained. They also cover the case in which $f(0) < 0$.

Theorem 0.4.3 (Farina-Sciunzi). *Let* $u \in C^2(\overline{\mathbb{R}^2_+})$ *be a solution of* (SPE+) *where* $f \in Lip_{loc}([0, +\infty))$. *Then,*

(i) if $f(0) \geq 0$, *then either* $u \equiv 0$, *or* u *is positive on* $\mathbb{R}^2_+$ *and strictly increasing in the direction orthogonal to the boundary, i.e.,* $\frac{\partial u}{\partial x_2} > 0$ *on* $\mathbb{R}^2_+$*;*

(ii) if $f(0) < 0$, *then either* $u > 0$ *and* $\frac{\partial u}{\partial x_2} > 0$ *on* $\mathbb{R}^2_+$, *or*

$$u(x) = u_0(x_2) \qquad \textit{for any } x \in \mathbb{R}^2_+$$

for a unique, periodic function $u_0 : [0, +\infty) \to [0, +\infty)$*;*

(iii) if for some $p > 1$,

$$\lim_{t \to +\infty} \frac{f(t)}{t^p} \in (0, +\infty), \tag{0.4.5}$$

then u *is bounded and one-dimensional.*[11]

(iv) if $f \in C^1([0, +\infty))$ *with* $f(0) < 0$ *and* $f'(t) \geq \alpha > 0$ *for any* $t > 0$, *then* u *is one-dimensional and periodic.*

Theorem 0.4.4 (Cortázar-Elgueta-García-Melián). *Assume* $N \geq 2$. *The only classical solution to*

$$\begin{cases} -\Delta u = u - 1 & \textit{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{in} \quad \mathbb{R}^N_+, \\ u = 0 & \textit{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

is the function

$$u(x) = 1 - \cos(x_N) \quad \textit{for any } x \in \mathbb{R}^N_+.$$

The case $N = 2$ in Theorem 0.4.4 is due to [FS16] (just apply item (iv) of Theorem 0.4.3 with $f(u) = u - 1$), while the case $N \geq 3$ is proven in [CEG16].

It is interesting to note that the one-dimensional symmetry results in the four theorems above were obtained without any additional assumption on the non-negative solutions u.

[11] more precisely, $u(x) = u_0(x_2)$ for any $x \in \mathbb{R}^2_+$ and for some bounded function $u_0 : [0, +\infty) \to [0, +\infty)$ solving (0.4.4). Moreover, either u_0 is periodic (possibly identically equal to zero) or, $u_0 > 0$ and $u_0' > 0$ on $(0, +\infty)$.

It therefore seems natural to seek to broaden the class of functions f guaranteeing that the only classical solutions, bounded or not, to (SPE+) are necessarily one-dimensional. On the other hand, such a one-dimensional symmetry result is not true for every locally Lipschitz-continuous function f, as shown by the function $u(x) = x_N e^{-x_1}$, which solves (SPE+) with $f(u) = -u$ and $N \geq 2$. Hence, the following two research lines naturally emerge :

(1) provide general natural conditions on f guaranteeing that the only classical solutions, bounded or not, to (SPE+) are necessarily one-dimensional;

(2) supply sufficiently general assumptions on classical solutions to (SPE+) ensuring their one-dimensional symmetry (either independently of the considered function $f \in Lip_{\text{loc}}([0, +\infty))$, or for an appropriate subclass of locally Lipschitz-continuous functions).

The new results presented in this section are devoted to both lines of research. Before stating our results, let us review those that exist in the literature and that relate to research line (2) above.

With regard to the second line of research, various additional hypotheses on u have been considered in the literature in order to establish one-dimensional symmetry of u. They mainly concern the growth conditions of u or its qualitative or spectral properties, such as monotonicity, stability and/or the Morse index. These conditions are natural, as they are generally satisfied by the solutions that appear in applications and by those constructed using variational or topological methods. In this regard, we present below the main contributions found in the literature. We shall describe them in ascending order of generality. Let us begin with the case of bounded solutions (see [BCN97a]).

Theorem 0.4.5 (Berestycki-Caffarelli-Nirenberg). *Assume $N = 2$ or 3. Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a positive and bounded solution to* (SPE+) *where $f \in Lip_{loc}([0, +\infty))$. If $N = 2$, u is one-dimensional and strictly increasing. If $N = 3$ the same conclusion holds, if one assumes in addition that $f(0) \geq 0$ and that f is C^1.*

More recently, by combining Theorem 0.4.3 above and a geometric tool introduced in [FV10], the authors of [FS16] improved the result of Theorem 0.4.5 in the two-dimensional case. Indeed, the authors of [FS16] prove the one-dimensional symmetry for any non-negative solution and assuming only that u has a bounded gradient. More precisely,

Theorem 0.4.6 (Farina-Sciunzi). *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to* (SPE+) *where $f \in Lip_{loc}([0, +\infty))$. If u has bounded gradient, then it is one-dimensional.*

Recall that, bounded solutions of (SPE+) have bounded gradient, but the converse is not true, as shown by the (one-dimensional) harmonic function $u(x) = x_N$. Furthermore, Theorem 0.4.6 also covers the case of one-dimensional periodic solutions which, given the homogeneous Dirichlet condition, must necessarily vanish somewhere in $\mathbb{R}^2_+$.[12]

To conclude on the case of bounded solutions of (SPE+) we mention the two following

[12] This fact is well illustrated by the explicit example discussed in Theorem 0.4.4.

results. The first one deals with convex functions f with $f(0) = 0$, while the second one applies, among other things, to any non-negative function f (see [CLZ14] and [DF22]).

Theorem 0.4.7 (**Chen-Lin-Zou**). *Assume $N \geq 2$ and let $f \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ with $f(0) = 0$ and $f'' \geq 0$. The only bounded classical solution to* (SPE+) *is $u \equiv 0$.*

As an immediate consequence of Theorem 0.4.7 one has

Corollary 0.4.1 (**Chen-Lin-Zou**). *Assume $N \geq 2$ and let $p > 1$. The only bounded classical solution to*

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

is $u \equiv 0$.

Corollary 0.4.1 was first proven for $1 < p < (N+1)/(N-3)^+$ in [Dan92], and then for any $1 < p < p_{JL}(N)$ in [Far07], where p_{JL} is the Joseph-Lundgren stability exponent given by $p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)^+}$.

Theorem 0.4.8 (**Dupaigne-Farina**). *Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution of* (SPE+). *Assume $f \in C^1([0, +\infty))$ and*

1. *either $f(t) \geq 0$ for $t \geq 0$,*
2. *or there exists $z > 0$ such that $f(t) \geq 0$ for $t \in [0, z]$ and $f(t) \leq 0$ for $t \geq z$.*

If $2 \leq N \leq 11$, then either $u \equiv 0$, or u is positive, one-dimensional and strictly increasing.

To continue, recall that a solution to (SPE+) that is either strictly monotone in the direction x_N or a local minimizer is automatically *stable*, which means that it additionally satisfies

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 - f'(u)\varphi^2 \geq 0, \tag{0.4.6}$$

for all $\varphi \in C^1_c(\mathbb{R}^N_+)$ (here we have supposed $f \in C^1$).
Also, we shall say that u is *stable outside a compact* $\mathcal{K} \subset \mathbb{R}^N_+$ if the integral inequality (0.4.6) holds for any $\varphi \in C^1_c(\mathbb{R}^N_+ \setminus \mathcal{K})$. Clearly, the stability implies the stability outside a compact set (actually, any finite Morse index solution to (SPE+) is stable outside a compact set of $\mathbb{R}^N_+$).

As for these classes of solutions to (SPE+), we have (see [DSS22])

Theorem 0.4.9 (**Dupaigne-Sirakov-Souplet**). *Assume $N \geq 2$. Let $f \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ be non-negative and convex function, with $f(0) = 0$ and $f \not\equiv 0$.*
Let u be a classical solution of (SPE+) *which is monotone in the x_N direction, i.e. such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Then, $u \equiv 0$.*
In particular, $u \equiv 0$ is the only classical solution to (SPE+) *which is bounded on finite strips.*[13]

[13] When $f(0) \geq 0$, any positive solution to (SPE+), which is also bounded on finite strips, is strictly

As an immediate consequence of Theorem 0.4.9 one has

Corollary 0.4.2 (**Dupaigne-Souplet-Sirakov**). *Assume $N \geq 2$ and let $p > 1$. Let u be a classical solution of*

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{0.4.7}$$

which is monotone in the x_N direction. Then, $u \equiv 0$.
In particular, $u \equiv 0$ is the only classical solution to (0.4.7) *which is bounded on finite strips.*

Corollary 0.4.2 recovers and improves upon Corollary 0.4.1 above.

Very recently, Corollary 0.4.2 has been improved in the work [DFP23]. Specifically,

Theorem 0.4.10 (**Dupaigne-Farina-Petitt**). *Assume $N \geq 2$ and let $p > 1$. The only classical solution to*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

which is stable outside a compact set of $\mathbb{R}^N_+$ is $u \equiv 0$.

Note that Theorem 0.4.10 holds for sign-changing solutions. It was first proven in [Far07] for all $N \leq 10$ and, if $N > 10$, for $1 < p < p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$, the Joseph-Lundgren stability exponent.

We are now ready to present our results. The first theorem provides a contribution to the first line of research above. It recovers and improves upon the result stated in item (iii) of the previous Theorem 0.4.3. It only requires that f behave at least linearly at infinity, i.e., that f satisfy (0.4.8) below.

Theorem 0.4.11 (**B.-Farina**). *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution of* (SPE+) *where $f \in Lip_{loc}([0, +\infty))$ satisfies*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{0.4.8}$$

Then there exists a bounded function $u_0 : [0, +\infty) \to [0, +\infty)$ such that

$$u(x) = u_0(x_2) \quad \textit{for any } x \in \mathbb{R}^2_+.$$

Moreover, either u_0 is periodic (possibly identically equal to zero) or, $u_0 > 0$ and $u_0' > 0$ on $(0, +\infty)$.

Next we state our results concerning the second line of research above. Let us begin with the case of solutions monotone in the x_N direction.

monotone in the x_N direction (see Theorem 3.1 of [Far20]).

Theorem 0.4.12 (B.-Farina). *Assume $2 \leq N \leq 9$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (SPE+) *such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Assume that $f \in C^1([0,+\infty))$ is a non-negative function satisfying*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0. \tag{0.4.9}$$

Then, either $u \equiv 0$, or there exists a bounded function $u_0 : [0,+\infty) \to [0,+\infty)$ such that

$$u(x) = u_0(x_N) \qquad \textit{for any } x \in \mathbb{R}^N_+$$

and $u_0 > 0$, $u_0' > 0$ on $(0,+\infty)$.
Moreover, if $f(t) > 0$ for any $t > 0$, then necessarily $u \equiv 0$ in $\mathbb{R}^N_+$ and $f(0) = 0$.

Although subject to a dimensional restriction, the previous result extends Theorem 0.4.9 to all non-negative functions f under the sole assumption that f behave at least linearly at infinity.[14] It would be interesting to know whether this result still holds for $N > 9$.

If we further assume that f is convex or that f has at most polynomial growth, the conclusions of Theorem 0.4.12 remain valid up to dimension $N = 10$. Specifically,

Theorem 0.4.13 (B.-Farina). *Assume $2 \leq N \leq 10$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (SPE+) *such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Assume that $f \in C^1([0,+\infty))$ is a non-negative, convex function satisfying*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0. \tag{0.4.10}$$

Then, either $u \equiv 0$, or there exists a bounded function $u_0 : [0,+\infty) \to [0,+\infty)$ such that

$$u(x) = u_0(x_N) \qquad \textit{for any } x \in \mathbb{R}^N_+$$

and $u_0 > 0$, $u_0' > 0$ on $(0,+\infty)$.
Moreover, if $f(0) = 0$, then necessarily $u \equiv 0$ in $\mathbb{R}^N_+$.

Unlike Theorem 0.4.7, the function f in Theorem 0.4.13 is not necessarily non-decreasing. For instance, Theorem 0.4.13 applies to $f(t) = |t-1|^p$, $p > 1$.

We also note that, if we drop the assumption (0.4.10), then the full conclusion of Theorem 0.4.13 would no longer be valid. Indeed, when $f(t) = e^{-2t}$, which is positive and convex, the *unbounded* and monotone function $u(x) = \ln(1 + x_N)$ solves (SPE+) for any $N \geq 2$.
This observation also applies to Theorem 0.4.12 above and to the following theorem.

Theorem 0.4.14 (B.-Farina). *Assume $2 \leq N \leq 10$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (SPE+) *such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Assume that $f \in C^1([0,+\infty))$ is a non-negative function satisfying*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0, \qquad \limsup_{t\to+\infty} \frac{f(t)}{t^\theta} < +\infty, \tag{0.4.11}$$

[14] Note that, the convex function f in Theorem 0.4.9 necessarily satisfies the assumption (0.4.9).

for some $\theta > 1$.
Then, either $u \equiv 0$, *or there exists a bounded function* $u_0 : [0, +\infty) \to [0, +\infty)$ *such that*

$$u(x) = u_0(x_N) \qquad \textit{for any } x \in \mathbb{R}^N_+$$

and $u_0 > 0$, $u_0' > 0$ *on* $(0, +\infty)$.
Moreover, if $f(t) > 0$ *for any* $t > 0$, *then necessarily* $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = 0$.

As an immediate consequence of previous three theorems we can treat the case of solutions that are bounded on finite strips.

Corollary 0.4.3 (**B.-Farina**). *Let* N, f *be as in the statement of Theorem 0.4.12 (resp. Theorem 0.4.13 or Theorem 0.4.14) and let* u *be a classical solution of* (SPE+) *which is bounded on finite strips.*
Then, the conclusions of Theorem 0.4.12 (resp. Theorem 0.4.13 or Theorem 0.4.14) still hold true.

The viewpoint we adopoted to prove the previous theorems can also be applied to both stable solutions and stable solutions outside a compact. Below we present the Liouville-type theorems corresponding to these two classes of solutions.

Theorem 0.4.15 (**B.-Farina**). *Assume* $2 \le N \le 4$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a stable solution of* (SPE+) *where* $f \in C^1([0, +\infty))$ *is a non-negative and non-decreasing function satisfying*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{0.4.12}$$

Then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = f'(0) = 0$.

In the next result we replace the superlinearity condition (0.4.12) on f with a growth assumption on the solution u.

Theorem 0.4.16 (**B.-Farina**). *Assume* $2 \le N \le 4$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a stable solution of* (SPE+) *where* $f \in C^1([0, +\infty))$ *is a non-negative, non-decreasing function with* $f \not\equiv 0$. *If*

$$u(x) = O(|x|^{\frac{4-N}{2}} \ln^{1/2} |x|) \qquad \textit{as} \quad |x| \to +\infty, \tag{0.4.13}$$

then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = f'(0) = 0$.

The following two results deal with solutions which are stable outside a compact set of $\mathbb{R}^N_+$.

Theorem 0.4.17 (**B.-Farina**). *Assume* $2 \le N \le 9$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (SPE+) *which is stable outside a compact set of* $\mathbb{R}^N_+$. *Let* $f \in C^1([0, +\infty))$ *be a non-negative, convex function with* $f(0) = 0$ *and* $f \not\equiv 0$.
Then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f'(0) = 0$.

Theorem 0.4.18 (**B.-Farina**). *Assume* $2 \le N \le 10$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (SPE+) *which is stable outside a compact set of* $\mathbb{R}^N_+$, *where* $f \in C^1([0, +\infty))$ *is a* *non-negative and non-decreasing function satisfying*

$$\lim_{t\to+\infty}\frac{f(t)}{t}=+\infty, \qquad \limsup_{t\to+\infty}\frac{f(t)}{t^{\theta}}<+\infty. \tag{0.4.14}$$

for some $\theta>1$.
Then, $u\equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0)=f'(0)=0$.

In the case of monotone solutions, the flexibility of our approach allows us to extend some of the previous results to the case of differential inequalities, even without imposing any boundary conditions. More precisely, we have

Theorem 0.4.19 (B.-Farina). *Assume* $N=2,3$ *and let* $u\in C^2(\mathbb{R}^N_+)$ *be a solution of*

$$\begin{cases} -\Delta u\geq f(u) & in \quad \mathbb{R}^N_+,\\ u\geq 0 & in \quad \mathbb{R}^N_+,\\ \frac{\partial u}{\partial x_N}\geq 0 & in \quad \mathbb{R}^N_+,\end{cases} \tag{0.4.15}$$

where $f\in C^0([0,+\infty))$ *satisfies* $f(t)>0$ *for any* $t>0$ *and*

$$\liminf_{t\to+\infty}\frac{f(t)}{t}>0.$$

Then, $u\equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0)=0$.

In particular, for $N=2,3$ and for any $p\geq 1$, the only non-negative and monotone solution to $-\Delta u\geq u^p$ in $\mathbb{R}^N_+$ is the function identically zero.

We also note that the result above is sharp. Indeed, the function $u(x',x_N)=c(1+|x'|^2)^{-\frac{1}{p-1}}$ satisfies $-\Delta u\geq u^p$ and $\frac{\partial u}{\partial x_N}\equiv 0$ in $\mathbb{R}^N_+$, when $N\geq 4$, $p>\frac{N-1}{N-3}$ and $c>0$ is small enough. Nevertheless, we have the following sharp Liouville-type result.

Theorem 0.4.20 (B.-Farina). *Assume* $N\geq 2$, *let* $p\geq 1$ *be a real number and let* $u\in C^2(\mathbb{R}^N_+)$ *be a solution of*

$$\begin{cases} -\Delta u\geq u^p & in \quad \mathbb{R}^N_+,\\ u\geq 0 & in \quad \mathbb{R}^N_+,\\ \frac{\partial u}{\partial x_N}\geq 0 & in \quad \mathbb{R}^N_+,\end{cases} \tag{0.4.16}$$

If $1\leq p\leq \bar{p}(N)$, *where*

$$\bar{p}(N):=\begin{cases} +\infty & if \quad N=2,3,\\ \frac{N-1}{N-3} & if \quad N\geq 4,\end{cases} \tag{0.4.17}$$

then, $u\equiv 0$ *in* $\mathbb{R}^N_+$.

If we restrict ourselves to solutions of the non-linear Poisson equation on $\mathbb{R}^N_+$, but still without imposing boundary conditions, the classification results above can be extended to dimension $N\geq 4$. This is the content of the following two theorems.

Theorem 0.4.21 (B.-Farina). *Assume $2 \le N \le 9$ and let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & in \quad \mathbb{R}^N_+, \\ u \ge 0 & in \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \ge 0 & in \quad \mathbb{R}^N_+, \end{cases} \tag{0.4.18}$$

where $f \in C^1([0, +\infty))$ satisfies $f(t) > 0$ for any $t > 0$ and

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{0.4.19}$$

Then, either $u \equiv 0$ in $\mathbb{R}^N_+$, or there exists a function $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ such that

$$u(x) = u_0(x_1, ..., x_{N-1}) \qquad \textit{for any } x \in \mathbb{R}^N_+.$$

If we further assume (0.4.11), the above conclusion holds true even in dimension $N = 10$.

In the case of the Lane-Emden equation $-\Delta u = u^p$, the above result can be improved as follows

Theorem 0.4.22 (B.-Farina). *Assume $N \ge 2$ and let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u = u^p & in \quad \mathbb{R}^N_+, \\ u \ge 0 & in \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \ge 0 & in \quad \mathbb{R}^N_+. \end{cases} \tag{0.4.20}$$

If $N = 2, 3$, then $u \equiv 0$ in $\mathbb{R}^N_+$; while for $N \ge 4$ we have:

(i) if $1 \le p < \frac{N+1}{N-3}$, then $u \equiv 0$ in $\mathbb{R}^N_+$;

(ii) if $p = \frac{N+1}{N-3}$, then either $u \equiv 0$ in $\mathbb{R}^N_+$ or

$$u(x) = (a + b|x' - x'_0|^2)^{-\frac{N-3}{2}} \qquad \textit{for any } x = (x', x_N) \in \mathbb{R}^N_+, \tag{0.4.21}$$

for $a, b > 0$ satisfying $1 = ab(N-1)(N-3)$ and any $x'_0 \in \mathbb{R}^{N-1}$;

(iii) if $\frac{N+1}{N-3} < p < p_{JL}(N-1)$, where $p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)^+}$ is the Joseph-Lundgren stability exponent, then either $u \equiv 0$ in $\mathbb{R}^N_+$, or there exists a function $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ such that

$$u(x) = u_0(x') \qquad \textit{for any } x = (x', x_N) \in \mathbb{R}^N_+,$$

When $N \ge 4$, for any $\frac{N+1}{N-3} < p < p_{JL}(N-1)$, there exists a positive solution to (0.4.20) depending only on the first $N-1$ variables.[15] Also note that, the positive solutions in items (*ii*) and (*iii*) are unstable (see Theorem 5 and Remark 4 of [Far07]). Finally observe

[15] For any $N \ge 3$ and $p > \frac{N+2}{N-2}$ positive radial solutions to $-\Delta u = u^p$ in $\mathbb{R}^N$ always exist (see for instance [GS81b],[JL73]). If u_0 is such a function in dimension $N-1$, then the function $u(x) = u_0(x')$, $x = (x', x_N) \in \mathbb{R}^N_+$, provides the desired example.

that, for $N \leq 11$, the result above provides the complete classification of the solutions to (0.4.20) for any $p \geq 1$.

In the remainder of this manuscript, the reader will find a summary of this thesis in French, followed by the chapters 1, 2 and 3, which respectively contain the article [BFS25], [BF25b] and [BF25a]. In these chapters, the reader will find not only the results set out in this introduction, but also other new results, together with their detailed proofs.

Perspectives

During this thesis, we have highlighted new geometric properties concerning possibly unbounded solutions of the following problem:

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{SPE}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a (locally or globally) Lipschitz-continuous function and $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an unbounded open set.

Thanks to our new comparison principles and the moving plane method adapted to the geometry of our domains, we proved new monotonicity results in various unbounded domains contained in a half-space.

These new results allowed us to obtain new classification results for possibly unbounded and non negative solutions to (SPE) in the half-space $\mathbb{R}^N_+$ with $N \leq 10$. We also studied solutions to the Serrin's overdetermined problem:

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{SOP}$$

in epigraphs bounded from below in all dimensions and provided partial answers to the following conjecture:

Conjecture D. *If f is a globally Lipschitz-continuous function and the overdetermined problem* (SOP) *admits a smooth and bounded solution on a smooth epigraph Ω, then Ω must be an affine half-space.*

There are still open questions and conjectures, and we have compiled a non-exhaustive list of these here:

1) In Theorem 0.1.1 in subsection 0.1.2, we assumes that $\mathfrak{c} \neq 0$. Is theorem 0.1.1 true if $\mathfrak{c} = 0$?

2) In section 0.4, we proved symmetry results for possibly unbounded non negative solutions to (SPE) in $\mathbb{R}^N_+$ for $N \leq 10$. *Can we prove symmetry results for possibly unbounded solutions to* (SPE) *in* $\mathbb{R}^N_+$*, in in dimensions higher than* 10?

3) Conjecture A and Conjecture D, as well as questions (0.1.1), remain open in dimensions higher than 3.

In view of our work, it seems interesting to know whether the tools we have developed can be used to study other open problems in mathematical literature, such as the following two conjectures:

In 1978, E. De Giorgi formulated his famous conjecture (see [De 78]):

De Giorgi's conjecture. *Let u be a bounded solution of the equation*

$$-\Delta u = u - u^3 \ \textit{in } \mathbb{R}^N,$$

such that $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N$. Then u is one dimensional, at least if $N \leq 8$.

In 1998, N. Ghoussoub and C. Gui (see [GG98]) solved De Giorgi conjecture in dimension 2 and, in 2000, L. Ambrosio and X. Cabré, in [AC00] proved De Giorgi's conjecture in dimension 3. For $N \leq 4$, the original conjecture is still open.

In [Sav09], O. Savin proves De Giorgi conjecture for $4 \leq N \leq 8$, under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1.$$

The second conjecture is due to E.N Dancer, who proposed the following stable solution conjecture (see [Dan10]):

Dancer's conjecture. *Assume $N \leq 7$ and let u be a bounded stable solution to*

$$-\Delta u = f(u) \ \textit{in } \mathbb{R}^N.$$

where f is a function of class C^1. Then u is one dimensional.

In [Dan10], E.N. Dancer originally states this conjecture for $N \leq 8$. Nevertheless, when $N \geq 8$, F. Pacard and J. Wei (see [PW13]) have build a bounded stable solution of the Allen-Cahn equation whose level sets are not hyperplanes. In view of this counterexample, Dancer's conjecture is now open only for $N \leq 7$.

Résumé en Français

Dans ce manuscrit, on étudie les propriétés des solutions de l'équation de Poisson semi-linéaire suivante:

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \Omega, \\ u > 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{SPE}$$

où $f : \mathbb{R} \mapsto \mathbb{R}$ est une fonction (localement ou globalement) Lipschitzienne et $\Omega \subset \mathbb{R}^N$ est un ensemble ouvert non borné ($N \geq 2$).

Ce problème, récurrent dans la littérature mathématique, est à l'origine de plusieurs conjectures et problèmes ouverts.

Dans une série d'articles (voir [BCN93], [BCN96], [BCN97a], [BCN97b]) H. Berestycki, L.A. Caffarelli et L. Nirenberg mettent en évidence de nombreuses propriétés géométriques des solutions du problème (SPE) et ont conjecturé plusieurs résultats.

Dans [BCN97a], ils s'intéressent aux solutions bornées de (SPE) dans le demi-espace $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N,\ x_N > 0\}$ et démontrent le résultat suivant:

Théorème 0.4.1 (Berestycki-Caffarelli-Nirenberg). *Dans le demi-espace, $\Omega = \mathbb{R}^N_+$, avec $N = 2$ ou 3, soit u une solution bornée de* (SPE)*. Si $N = 2$, alors u est symétrique (i.e. $u = u(x_N)$). Si $N = 3$, alors la conclusion reste vraie, si on suppose que $f(0) \geq 0$ et que f est de classe C^1.*

Afin de prouver ce théorème, les auteurs montrent que $f(\sup u) = 0$ et, appliquent ensuite le théorème suivant (voir [BCN93]):

Théorème 0.4.2 (Berestycki-Caffarelli-Nirenberg). *Soit $u \in C^2(\mathbb{R}^N_+)$ une solution bornée de* (SPE) *dans le demi-espace, avec f une fonction localement Lipschitzienne qui satisfait:*

$$f(\sup u) \leq 0.$$

Alors u est strictement croissante dans la direction x_N (i.e, $\frac{\partial u}{\partial x_N} > 0$ dans $\mathbb{R}^N_+$) et u dépend seulement de la variable x_N et $f(\sup u) = 0$.

Au vu de ces résultats, H. Berestycki, L.A. Caffarelli et L. Nirenberg sont amenés à faire la conjecture suivante:

Conjecture A. *S'il existe une solution bornée u de* (SPE) *avec* $\Omega = \mathbb{R}^N_+$ *alors* $f(\sup u) = 0$ *et* (*en appliquant le résultat 0.4.2*) *u est symétrique et monotone dans la direction x_N* (*i.e,* $\frac{\partial u}{\partial x_N} > 0$) .

À cet égard, A. Farina et E. Valdinoci, dans [FV10], prouvent un résultat de symétrie unidimensionnelle pour des solutions bornées de (SPE) dans $\mathbb{R}^N_+$, avec $N \leq 5$ et pour des fonctions $f \in C^1([0, +\infty))$ satisfaisant l'une des assertions suivantes:

(a) $f(t) \geq 0$, pour tout $t \geq 0$,

(b) Il existe $\zeta > 0$, tel que $f(t) \geq 0$ pour tout $t \in [0, \zeta]$ et $f(t) \leq 0$ pour tout $t \in [\zeta, +\infty)$.

L. Dupaigne et A. Farina ([DF22]) ont étendu ce résultat jusqu'à la dimension 11 (voir le théorème 0.8.8). En exploitant (a) et (b), les auteurs prouvent que $f(\sup u) = 0$ (et donc la conjecture A), et concluent alors en appliquant le théorème 0.4.2 ci-dessus.

Dans [BCN97a], les auteurs testent la conjecture A en prenant $f(u) = u - 1$ et ils remarquent que dans ce cas la conjecture est fausse. En effet, ils observent que s'il existe une solution u de (SPE) avec $\Omega = \mathbb{R}^N_+$ et $f(u) = u - 1$ alors $\sup u > 2$, or $f(t) > 0$ pour tout $t \geq 2$. Ceci les conduit à formuler la conjecture suivante:

Conjecture B. *Il n'existe pas de solution bornée au problème suivant:*

$$\begin{cases} -\Delta u = u - 1 & \textit{dans} \quad \mathbb{R}^N_+, \\ u > 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ u = 0 & \textit{sur} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{0.4.22}$$

Dans [BCN97a], H. Berestycki, L.A. Caffarelli et L. Nirenberg démontrent cette conjecture en dimension 2 et 3 (voir Proposition 1.6 dans [BCN97a]). En 2016, dans [CEG16], C. Cortázar, M. Elgueta et J. Garcia-Melián résolvent complètement la conjecture B. Précisément, les auteurs démontrent le résultat suivant:

Théorème 0.4.3 (Cortázar-Elgueta-Garciá-Melian). *Supposons $N \geq 2$. L'unique solution, potentiellement non bornée, du problème*

$$\begin{cases} -\Delta u = u - 1 & \textit{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ u = 0 & \textit{sur} \quad \partial\mathbb{R}^N_+, \end{cases}$$

est la fonction

$$u(x) = 1 - \cos(x_N) \quad \textit{pour tout } x \in \mathbb{R}^N_+.$$

Ce résultat a d'abord été démontré par A. Farina et B. Sciunzi en dimension 2 (voir [FS16].

Le résultat ci-dessus suggère que les solutions positives au problème de Dirichlet homogène pour les équations de Poisson semi-linéaires doivent également être prises en compte. Pour cette raison, dans cette thèse, nous étudions également des solutions positives et éventuellement non bornées au problème suivant (voir la sous-section 0.6.3 et la section 0.8):

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \Omega, \\ u \geq 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega. \end{cases} \tag{SPE+}$$

Jusqu'à présent les résultats et conjectures énoncés le sont dans le demi-espace et une question naturelle se pose: *Que ce se passe t-il lorsque la géométrie de Ω est plus complexe?*

Une généralisation naturelle du demi-espace est donnée par le cas d'une épigraphe, c'est-à-dire où Ω a la forme suivante:

$$\Omega := \{(x', x_N) \in \mathbb{R}^N, x_N > g(x')\}, \tag{0.4.23}$$

où $g : \mathbb{R}^{N-1} \to \mathbb{R}$ est une fonction continue.

Pour de tels domaines, M.J. Esteban and P.L. Lions, dans leur article [EL82], prouvent le résultat de monotonie suivant:

Théorème 0.4.4 (Esteban-Lions). *Soit $g : \mathbb{R}^{N-1} \to \mathbb{R}$ une fonction régulière et coercive, c'est-à-dire, qui satisfait*

$$\lim_{|x'| \to +\infty} g(x') = +\infty, \tag{0.4.24}$$

et Ω son épigraphe (voir (0.4.23)). Si $u \in C^2(\overline{\Omega})$ est une solution de (SPE) avec $f \in Lip_{loc}([0, +\infty))$ et $f(0) \geq 0$, alors

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{dans } \Omega.$$

C'est le premier résultat de monotonie établit dans un épigraphe, et la question est de savoir si le théorème 0.4.4 reste vrai quand l'épigraphe Ω n'est plus supposé coercif et/ou quand $f(0) < 0$. Dans ce manuscrit, on s'intéresse également à cette question.

Afin de répondre à une question de R.L. Fosdick concernant la mécanique des fluides, J. Serrin est amené, dans son article fondateur publié en 1971 [Ser71], à considérer les solutions régulières et bornées du problème sur-déterminé suivant:

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \Omega, \\ u > 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{sur} \quad \partial\Omega, \end{cases} \tag{SOP}$$

où Ω est un domaine lisse et borné et η désigne le vecteur unitaire normal à $\partial\Omega$ et f est une fonction Lipschitzienne. Concernant ce problème, J. Serrin prouve son célèbre résultat:

Théorème 0.4.5 (**Serrin**). *Soit* Ω *un domaine bornée de classe* C^2*. Supposons qu'il existe une fonction* $u \in C^2(\overline{\Omega})$ *satisfaisant* (SOP)*. Alors* Ω *est une boule et* u *a la forme suivante:*

$$u(x) = \frac{b^2 - |x|^2}{2N},$$

où b *est le rayon de la boule et* $|x|$ *désigne la distance au centre de la boule.*

Afin de prouver ce résultat, J. Serrin utilise, pour la première fois, la méthode des hyperplans mobiles ("moving plane method" en anglais). Cette méthode est basée sur un principe de réflexion de A. D. Alexandrov (voir [Ale62]).

Dans [BCN97b], H. Berestycki, L.A. Caffarelli et L. Nirenberg s'intéressent au problème (SOP) dans le cas cas où Ω est un épigraphe régulier (voir 0.4.23) et f est une fonction de type Allen-Cahn. Les auteurs emploient la méthode du glissement ("sliding method" en anglais) afin de prouver le résultat de rigidité suivant:

Théorème 0.4.6 (**Berestycki-Caffarelli-Nirenberg**). *Soit* f *une fonction de type Allen-Cahn. Soit* $g : \mathbb{R}^{N-1} \to \mathbb{R}$ *une fonction de classe* C^2 *qui satisfait la condition suivante:*

$$\textit{Pour tout } \tau \in \mathbb{R}^{N-1} \qquad \lim_{x \in \mathbb{R}^{N-1}, |x| \to \infty} g(x+\tau) - g(x) = 0, \tag{0.4.25}$$

et Ω *son épigraphe. S'il existe une solution bornée* u *au problème* (SOP) *alors* Ω *est un demi-espace et* u *est symétrique et monotone.*

À la suite de ce résultat, les auteurs formulent la question suivante:

Conjecture C. *Supposons que* Ω *est un domaine lisse tel que* Ω^c *(le complémentaire de* Ω*) est connexe et qu'il existe une solution bornée de* (SOP) *où* f *est une fonction localement Lipschitzienne. Alors* Ω *est soit un demi-espace, soit une boule, soit le complémentaire d'une boule ou d'un cylindre de la forme* $\mathbb{R}^j \times B$ *avec* B *une boule.*

Cette question est connue sous le nom de "Conjecture de Berestycki-Caffarelli-Nirenberg". Motivés par le théorème 0.4.6, nous abordons la conjecture C dans le cas particulier où Ω est un épigraphe, ce qui conduit à la variante de la conjecture D suivante :

Conjecture D. *Si* f *est une fonction globalement Lipschitzienne et si le problème sur-déterminé* (SOP) *possède une solution régulière et bornée définie sur un épigraphe régulier* Ω*, alors* Ω *doit être un demi-espace affine.*

Toutes ces conjectures nous ont poussées à étudier les propriétés qualitatives et géométriques des solutions de (SPE) dans des domaines non-bornés variés. Dans le reste de cette introduction, on présente de manière unifiée, les résultats obtenus durant cette thèse que l'on compare avec les résultats existant dans la littérature. Tout ces résultats sont contenus dans 3 articles de recherche qui sont respectivement l'objet des chapitres 1, 2 et 3 (voir [BFS25], [BF25b], [BF25a]).

Les prochaines sections s'organisent comme suit:

1) Dans la première section (voir 0.5), on introduit nos nouveaux résultats de monotonie. Dans la première sous-section, on décrit les résultats de monotonie pour des solutions potentiellement non bornées de (SPE) dans des épigraphes minorés et pour des fonctions f localement ou globalement Lipschitzienne satisfaisant $f(0) \geq 0$ (voir la sous-section 0.5.1). La seconde sous-section est dédiée à la présentation de nos résultats de monotonie sans restriction sur le signe de $f(0)$ (voir la sous-section 0.5.2). Dans ce cadre, les résultats sont établis pour des solutions, potentiellement non bornées de (SPE) ou (SOP) dans des domaines non bornés variés. Dans la troisième sous-section, on présente des extensions de nos résultats de monotonie à une vaste classe d'épigraphes continus, minorés (voir la sous-section 0.5.3). La démonstration de tous ces résultats repose sur de nouveaux principes de comparaison dans des domaines non bornés. Nous présenterons et décrirons ces nouveaux principes dans la deuxième section ci-dessous. Finalement, dans le dernière sous-section, on donne les premières applications de nos résultats de monotonie sur la classification des solutions bornées de (SPE+) dans des épigraphes minorés et en toutes dimensions (voir la sous-section 0.5.4).

2) Dans la section 0.6, nous établissons de nouveaux principes de comparaison (voir la sous-section 0.6.2) pour une grande classe de sous-ensembles euclidiens non bornés, que nous appelons *ensembles avec des bonnes sections* (voir la sous-section 0.6.1). Ces principes de comparaison servent à initialiser et à compléter une version modifiée de la méthode des hyperplans mobiles, adaptée à la géométrie de l'épigraphe Ω. Ceci est crucial pour démontrer nos résultats de monotonie décrits dans la première section. Enfin, nos nouveaux principes de comparaison nous permettent de prouver quelques nouveaux résultats d'unicité et de symétrie pour des solutions, éventuellement non bornées et pouvant changer de signe, au problème de Dirichlet homogène pour l'équation de Poisson semi-linéaire dans des domaines non bornés généraux (voir la sous-section 0.6.3).

3) Dans la troisième section (voir 0.3), nous démontrons la validité de la conjecture D dans plusieurs cas. Plus précisément, nous considérons le problème sur-déterminé de Serrin (SOP) pour une famille de non-linéarités spécifiques de type Allen-Cahn. Dans ce cadre, nous démontrons la validité de la conjecture D en toute dimension $N \geq 2$, pour toute solution de (SOP) et pour une grande classe d'épigraphes minorés (voir la sous-section 0.7.1). Enfin, nous considérons les solutions de (SOP) dans des épigraphes réguliers et minorés où f est une fonction localement ou globalement Lipschitzienne. Dans ce cadre, nous prouvons la conjecture D avec $N \leq 3$, pour des solutions u éventuellement non bornées (voir la sous-section 0.7.2).

4) La dernière section (voir section 0.4) est consacrée à la présentation de nouveaux résultats de classification pour les solutions potentiellement non bornées de (SPE+) dans le demi-espace. Nos résultats ont été obtenus en suivant les deux axes de recherche ci-dessous :

   (I) fournir des conditions naturelles générales sur f garantissant que les seules solutions classiques, bornées ou non, de (SPE+) sont nécessairement unidimensionnelles.

(II) Fournir des hypothèses suffisamment générales sur les solutions classiques de (SPE+) garantissant leurs symétries unidimensionnelles.

0.5 Résultats de monotonie dans des domaines non bornés

Dans cette section, nous présentons nos nouveaux résultats de monotonie pour les solutions (bornées ou non) de (SPE) définies dans des domaines non bornés inclus dans un demi-espace, ainsi que pour des non-linéarités (localement ou globalement) Lipschitziennes. En particulier, nous nous intéressons au cas où Ω est un épigraphe minoré, c'est-à-dire,

$$\Omega := \{(x', x_N) \in \mathbb{R}^N,\ x_N > g(x')\},$$

où $g : \mathbb{R}^{N-1} \to \mathbb{R}$ est une fonction continue et minorée.

On rappelle que, lorsque $f(0) \geq 0$, le principe du maximum affirme qu'une solution de (SPE+) dans tout domaine Ω est soit identiquement égale à zéro, soit strictement positive et que cette propriété n'est plus vraie lorsque $f(0) < 0$ (voir l'exemple donné par $f(u) = u - 1$ et le théorème 0.4.3). Au vu de cette observation, nous avons décidé, dans cette section, de présenter d'abord les résultats dans le cas $f(0) \geq 0$, et ensuite ceux valables quel que soit le signe de $f(0)$. Le cas $f(0) \geq 0$ est discuté dans la sous-section 0.5.1, tandis que les résultats de monotonie qui sont valables quel que soit le signe de $f(0)$ sont situés dans la sous-section (0.5.2).

La sous-section 0.5.3 est consacrée à présenter quelques extensions de nos résultats de monotonie à une large classe d'épigraphes minorés.

Enfin, dans la dernière sous-section (voir sous-section 0.5.4), nous décrivons une première application de nos nouveaux résultats de monotonie. Il s'agit de la classification des solutions bornées de (SPE+) dans des épigraphes minorés.

Dans chaque sous-section, nous présentons d'abord les résultats existants dans la littérature mathématique, puis nos nouveaux résultats sur la monotonie.

0.5.1 Résultats de monotonie lorsque $f(0) \geq 0$

Le cas où $f(0) \geq 0$ est le plus étudié dans la littérature mathématique. Le premier résultat à cet égard est dû à E.N. Dancer ([Dan92]). Dans cet article, l'auteur étudie le problème (SPE) dans le demi-espace et, grâce à la méthode des hyperplans mobiles, démontre le théorème suivant:

Théorème 0.5.1 (Dancer). *Supposons que f est de classe C^1, avec $f(0) > 0$ ou $f(0) = 0$ et $f'(0) \geq 0$. Soit $u \in C^2(\overline{\mathbb{R}^N_+})$ une solution bornée de* (SPE) *dans le demi-espace. Alors $\frac{\partial u}{\partial x_N}(x) > 0$ si $x \in \overline{\mathbb{R}^N_+}$.*

Quelques années plus tard, H. Berestycki, L.A. Caffarelli et L. Nirenberg, dans [BCN97a], démontrent le résultat suivant (voir corollaire 1.3 dans [BCN97a]):

Théorème 0.5.2 (Berestycki-Caffarelli-Nirenberg). *Dans le demi-espace $\Omega = \mathbb{R}^N_+$, supposons que u est une solution de* (SPE) *où f est une fonction globalement Lipschitzienne avec $f(0) \geq 0$. Alors, la fonction u satisfait*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{dans } \Omega.$$

Ce résultat s'applique aux solutions potentiellement non bornées de (SPE) avec $f(0) \geq 0$, et recouvre le théorème 0.5.1. Cependant, les auteurs supposent que f est une fonction globalement lipschitzienne. Par conséquent, le théorème 0.5.2 ne s'applique pas aux fonctions telles que $f(t) = t^p$ $(p > 1)$ ou $f(t) = t - t^3$ (qui donnent respectivement l'équation de Lane-Emden et l'équation d'Allen-Cahn). De ce fait, les mêmes auteurs, dans [BCN96], soulèvent la question suivante:

Problème ouvert 0.5.1. La conclusion du théorème (0.5.2) est-elle toujours valable pour les solutions non bornées de (SPE) dans le cas où f est localement Lipschitzienne?

La preuve du théorème 0.5.2 repose sur la méthode des hyperplans mobiles. Pour démarrer cette procédure, les auteurs utilisent un de leurs résultats établis dans [BCN96], où ils démontrent que les solutions positives de (SPE) dans $\mathbb{R}^N_+$ ont une croissance exponentielle sur des bandes finies [16]. Ils utilisent notamment le caractère lipschitzien de f.

Dans [Far20], A. Farina considère également le problème (SPE) dans le demi-espace où f est localement Lipschitzienne et donne, ainsi, une première réponse au problème ouvert (0.5.1). En effet, il démontre le résultat suivant:

Théorème 0.5.3 (Farina). *Soient $N \geq 2$, $\Omega = \mathbb{R}^N_+$, $f \in Lip_{loc}([0, +\infty))$ avec $f(0) \geq 0$ et soit $u \in C^2(\overline{\mathbb{R}^N_+})$ une solution de* (SPE)*. Supposons que u est bornée sur les bandes finies $\mathbb{R}^{N-1} \times [0, t]$, pour tout $t > 0$, c'est-à-dire,*

$$\forall t > 0, \ \exists C(t) > 0 \,, \textit{tel que } 0 < u \leq C(t) \quad \textit{dans } \mathbb{R}^{N-1} \times [0, t].$$

Alors u est strictement croissante dans la direction x_N, i.e, $\frac{\partial u}{\partial x_N} > 0$ dans $\mathbb{R}^N_+$.

Ce résultat s'applique à toutes les fonctions localement lipschitziennes tel que $f(t) = t - t^3$ ou $f(t) = t^p$, mais il demande que u soit bornée sur des bandes finies.

Tous les résultats ci-dessus sont démontrés en exploitant de manière cruciale le fait que $\Omega = \mathbb{R}^N_+$. Il semble donc intéressant de savoir ce qui se passe lorsque Ω est un domaine non borné différent de $\mathbb{R}^N_+$.

Dans ce contexte, les résultats sont peu nombreux et le premier résultat en ce sens est dû à M.J. Esteban et P.L. Lions qui, en 1982, se sont intéressés au problème (SPE) dans

[16] i.e. u satisfait la propriété suivante: pour tout $b > 0$, il existe α, $\beta > 0$ dépendant de b, tel que

$$u(x, y) \leq \alpha e^{\beta|x|} \quad \text{pour tout } (x, y) \in \mathbb{R}^{N-1} \times [0, b].$$

un domaine non borné particulier (voir [EL82]).

Théorème 0.5.4 (Esteban-Lions). *Soit Ω un domaine régulier satisfaisant les conditions suivantes:*

(i) Si $x = (x_1, \cdots, x_N) \in \Omega$ alors $y = (y_1, \cdots, y_N) \in \Omega$, pour tout $y_N \geq x_N$,

(ii) $m = \inf\{x_N |$ il existe $x' \in \mathbb{R}^{N-1}$ tel que $x = (x', x_N) \in \Omega\} > -\infty$,

(iii) pour tout $\lambda > 0$, l'ensemble $\Sigma(\lambda) := \{x \in \Omega | \, x_N < \lambda\}$ est borné .

Si $u \in C^2(\overline{\Omega})$ est une solution de (SPE) *avec $f \in Lip_{loc}([0, +\infty))$ et $f(0) \geq 0$, alors*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{dans } \Omega.$$

Il est facile de voir que si Ω est un épigraphe coercif, alors les hypothèses (i), (ii) et (iii) sont satisfaites.

Plus tard, dans [BCN97b], H. Berestycki, L.A. Caffarelli et L. Nirenberg traitent du problème (SPE) dans un épigraphe globalement Lipschitzien régulier où f est une fonction de type Allen-Cahn, c'est-à-dire :

Définition 0.5.1. $f : [0, +\infty) \to \mathbb{R}$ est une fonction de type Allen-Cahn si f est localement Lipschitzienne et satisfait les assertions suivantes:

$$\begin{cases} & \exists\, \mu > \delta_2 > \delta_1 > 0 \ : \\ \text{(I)} & f > 0 \quad \text{dans } (0, \mu), \qquad f \leq 0 \quad \text{dans } [\mu, +\infty), \\ \text{(II)} & f(t) \geq \delta_1 t \quad \text{pour tout } t \in (0, \delta_1), \qquad f \text{ est décroissante dans } (\delta_2, \mu). \end{cases} \tag{0.5.1}$$

Le prototype de fonction satisfaisant la définition précédente est $f(t) = t - t^3$, qui donne l'équation d'Allen-Cahn $-\Delta u = u - u^3$ (voir [AC79]), d'où le nom de "fonction de type Allen-Cahn". Pour de telles fonctions, les auteurs démontrent le théorème suivant:

Théorème 0.5.5 (Berestycki-Caffarelli-Nirenberg). *Soit Ω un épigraphe globalement Lipschitzien et $f : [0, +\infty) \to \mathbb{R}$ une fonction de type Allen-Cahn. Si $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ est une solution bornée de* (SPE)*, alors u est strictement croissante dans la direction x_N, c'est-à-dire,*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{dans } \Omega.$$

Ce résultat s'applique à tout épigraphe globalement Lipschitzien, cependant la preuve est basée de manière cruciale sur la forme de f, qui est de type Allen-Cahn ainsi que sur le caractère borné de u.

Pour prouver ce résultat, les auteurs utilisent la méthode du glissement ("sliding method" en Anglais) (voir le lemme 3.1 dans [BCN97b]) qui consiste à montrer que si deux fonctions u et z sont ordonnées dans une boule B telle que $\overline{B}$ est contenue dans Ω et qu'elles vérifient les propriétés suivantes:

$$\begin{cases} -\Delta u \geq f(u) & \text{dans} & \Omega, \\ -\Delta z \leq f(z) & \text{où } z > 0 \text{ dans } B, & \\ z \leq u & \text{dans} & B, \\ z \leq 0 & \text{sur} & \partial B. \end{cases}$$

Alors, pour toute famille continue de mouvements euclidiens à un paramètre (c'est-à-dire de translations et de rotations) $A(t)$ pour $0 \leq t \leq T$ avec $A(0) = Id$ et $A(t)\overline{B} \subset D$ pour tout t, nous avons pour tout $t \in [0, T]$:

$$z_t(x) := z(A(t)^{-1}x) < u(x) \text{ dans } B_t := A(t)B.$$

Nous sommes maintenant en mesure d'énoncer nos nouveaux résultats de monotonie obtenus pour des épigraphes minorés et pour des fonctions f localement ou globalement Lipschitziennes. Commençons par le cas des épigraphes uniformément continus:

Théorème 0.5.6 (B.-Farina-Sciunzi). *Soit Ω un épigraphe uniformément continu et minoré. Supposons $f \in Lip_{loc}([0, +\infty))$ avec*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0,$$

et soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de (SPE) *qui est bornée et uniformément continue sur des bandes finies.*[17] *Alors u est strictement croissante dans la direction x_N, c'est-à-dire que $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*[18]
Si f est globalement Lipschitzienne, la même conclusion est vraie sous la seule hypothèse que u est uniformément continue sur des bandes finies.

Les épigraphes considérés sont très généraux, par exemple si l'on prend l'épigraphe d'une fonction de Weierstrass, c'est-à-dire un épigraphe défini par

$$g_{b,\alpha}(x) := \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n \pi x),$$

où $b > 1$ est un entier et $\alpha \in (0, 1)$, alors notre résultat est vrai pour une solution de (SPE) définie dans cet épigraphe, car $g_{b,\alpha}$ est une fonction uniformément continue, bornée, mais nulle part différentiable (voir [Har16]) donc elle n'est même pas localement Lipschitzienne.

La preuve du théorème 0.5.6 repose en partie sur la méthode des hyperplans mobiles adaptée aux épigraphes. Pour initialiser cette méthode, nous devons exiger que les épigraphes soient minorés.

Si nous supposons en outre que l'épigraphe satisfait une hypothèse de régularité (voir la définition (0.5.2)), nous pouvons prouver le résultat de monotonie pour toutes fonctions f

[17] c'est-à-dire, pour tout $R > 0$, u est bornée et uniformément continue sur la bande $\Omega \cap \{x_N < R\}$

[18] Les solutions distributionnelles continues de $-\Delta u = f(u)$ dans Ω appartiennent à $C^2(\Omega)$, selon la théorie elliptique standard.

(localement ou globalement) Lipschitziennes satisfaisant $f(0) \geq 0$ (et cela en affaiblissant également les hypothèses sur u).

Théorème 0.5.7 (B.-Farina-Sciunzi). *Soit Ω un épigraphe uniformément continu et minoré satisfaisant la condition du cône extérieur uniforme.*
Supposons $f \in Lip([0, +\infty))$ avec $f(0) \geq 0$ et soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de (SPE) *avec, au plus, une croissance exponentielle sur les bandes finies, i.e., pour tout $R > 0$, il existe deux réels positifs $A = A(R), B = B(R)$ tel que:*

$$u(x) \leq Ae^{B|x|} \qquad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.

Nous pouvons maintenant définir la condition du cône extérieur uniforme:

Définition 0.5.2 (Condition du cône extérieur uniforme). Un ouvert $\omega \subset \mathbb{R}^N$ (pas nécessairement un épigraphe) vérifie une condition de cône extérieur uniforme si, pour tout $x_0 \in \partial\omega$, il existe un cône fini V_{x_0}, de sommet x_0, tel que $\overline{\omega} \cap V_{x_0} = \{x_0\}$ et que les cônes V_{x_0} soient tous congrus à un cône fixe V. Le cône V est appelé cône de référence.

Il est facile de voir que tous les épigraphes globalement Lipschitziens vérifient cette propriété et que, dans ce cas, le cône de référence dépend de la constante de Lipschitz de la fonction définissant l'épigraphe. Cependant, l'inverse n'est pas vrai comme le montrent les exemples suivants. L'épigraphe défini par la fonction $x \mapsto e^x$ vérifie la condition de cône extérieur uniforme (par convexité), mais n'est pas uniformément continu. L'épigraphe définie par la fonction g ci-dessous, est $\frac{1}{2}$-Hölderien et satisfait la condition de cône extérieur uniforme mais il n'est pas globalement Lipschitzien (voir la figure (5)).

$$g(x) = \begin{cases} 0 & \text{si} \quad x \in (-\infty, -4], \\ \sqrt{4 - (x+2)^2} & \text{si} \quad x \in [-4, 0], \\ \sqrt{4 - (x-2)^2} & \text{si} \quad x \in [0, 2], \\ 2 & \text{si} \quad x \in [2, +\infty), \end{cases}$$

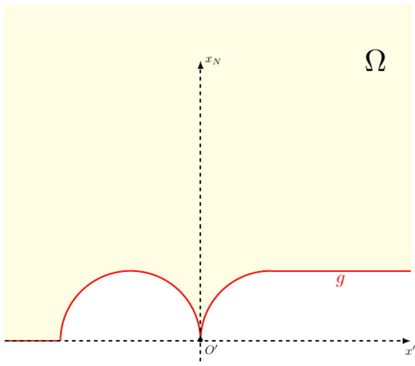

Figure 5: g

Notons que l'on retrouve ici le théorème 0.5.2 de H. Berestycki, L.A. Caffarelli et L. Nirenberg dans le demi-espace, puisque dans ce cas, comme nous l'avons déja dit la solution u croît exponentiellement sur les bandes finies (voir [BCN96]).

Nous étudions également les solutions de (SPE) dans un épigraphe uniformément continu minoré lorsque f est une fonction localement Lipschitzienne et nous prouvons le résultat suivant:

Théorème 0.5.8 (**B.-Farina-Sciunzi**). *Soit Ω un épigraphe uniformément continu et minoré satisfaisant la condition du cône extérieur uniforme. Supposons $f \in Lip_{loc}([0,+\infty))$ avec $f(0) \geq 0$ et soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de* (SPE) *qui est bornée sur les bandes finies. i.e., pour tout $R > 0$,*

$$\sup_{\Omega \cap \{x_N < R\}} u \; < +\infty.$$

Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.

Si $\Omega = \mathbb{R}^N_+$ alors nous retrouvons le théorème 0.5.3 de A. Farina puisque, $\mathbb{R}^N_+$ est un épigraphe uniformément continu et minoré et il satisfait la condition du cône extérieur uniforme.

Notons que le théorème 0.5.7 et le théorème 0.5.8 couvrent le cas des épigraphes uniformément continus minorés, qui ne sont pas nécessairement globalement Lipschitzien. En effet, le théorème 0.5.8 s'applique à l'épigraphe définit par la fonction g ci-dessous (voir la figure 6):

$$g(x) = \begin{cases} 0 & \text{si} \quad x \in (-\infty, -4], \\ \sqrt{4-(x+2)^2} & \text{si} \quad x \in [-4, 0], \\ \sqrt{4-(x-2)^2} & \text{si} \quad x \in [0, 2], \\ 2 & \text{si} \quad x \in [2, 6], \\ x-4 & \text{si} \quad x \in [6, +\infty). \end{cases}$$

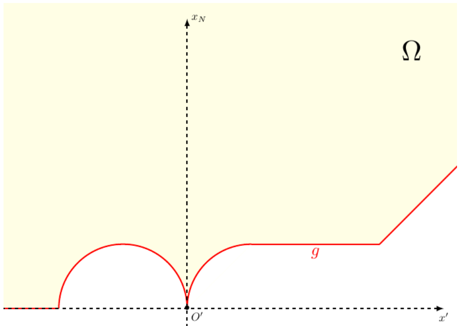

Figure 6: g

Notons que g est uniformément continue, minorée, mais n'est pas localement Lipschitzienne. De plus, on peut voir qu'elle satisfait la condition du cône extérieur uniforme.

Nous allons maintenant présenter quelques corollaires directs des résultats précédents, qui sont nouveaux même dans leur forme plus simple.

Corollaire 0.5.1 (**B.-Farina-Sciunzi**). *Soit Ω un épigraphe uniformément continu, minoré. Supposons que $f \in Lip_{loc}([0,+\infty))$ avec*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0,$$

et soit $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ une solution classique de (SPE) *telle que $\nabla u \in L^\infty(\Omega)$. Alors u est strictement croissante dans la direction x_N, c'est-à-dire que $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*

Corollaire 0.5.2 (**B.-Farina-Sciunzi**). *Soit Ω un épigraphe globalement Lipschitzien minoré. Supposons $f \in Lip_{loc}([0,+\infty))$ avec $f(0) \geq 0$ et soit $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ une solution classique de* (SPE) *telle que $\nabla u \in L^\infty(\Omega)$. Alors u est strictement croissante dans la direction x_N, c'est-à-dire que $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*

Corollaire 0.5.3 (**B.-Farina-Sciunzi**). *Supposons* $\alpha \in (0,1)$ *et soit* Ω *un épigraphe globalement Lipschitzien minoré avec* $g \in C^{1,\alpha}_{loc}(\mathbb{R}^{N-1})$. *Supposons* $f \in Lip_{loc}([0,+\infty))$ *avec* $f(0) \geq 0$ *et soit* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *une solution classique de* (SPE) *qui est bornée sur des bandes finies. Alors* u *est strictement croissante dans le direction* x_N*, i.e.,* $\frac{\partial u}{\partial x_N} > 0$ *dans* Ω.

0.5.2 Résultats de monotonie indépendament du signe de $f(0)$

Dans cette sous-section, nous présentons des résultats de monotonie valables quelque soit le signe de $f(0)$. En raison de l'absence du principe du maximum fort et du lemme de Hopf lorsque $f(0) < 0$, très peu de résultats sont connus.

De plus, lorsque $f(0) < 0$, il peut exister des solutions supérieures ou égales à zéros de (SPE) qui ne sont pas croissantes dans la direction x_N et ce sont parfois les seules, comme le montre $f(u) = u - 1$ (voir [FS16], [BCN97a], [CEG16], [FS13]).

Le premier résultat que nous citons est un résultat de monotonie et de symétrie dû à H. Berestycki, L.A. Caffarelli et L. Nirenberg. En effet, dans [BCN93], ils démontrent le théorème suivant:

Théorème 0.5.9 (**Berestycki-Caffarelli-Nirenberg**). *Soit* $u \in C^2(\mathbb{R}^N_+)$ *une solution bornée de* (SPE) *dans le demi-espace, avec* f *une fonction localement Lipschitzienne qui satisfait:*

$$f(\sup u) \leq 0.$$

Alors u *est strictement croissante dans la direction* x_N *(i.e,* $\frac{\partial u}{\partial x_N} > 0$ *dans* $\mathbb{R}^N_+$*) et* u *dépend seulement de la variable* x_N *et* $f(\sup u) = 0$.

L'hypothèse sur le caractère borné de u est optimale, par exemple, la fonction croissante $u(x_1, \cdots, x_N) = x_N e^{x_1}$ est solution (SPE) dans le demi-espace avec $f(u) = -u$.

Dans ce contexte, H. Berestycki, L.A. Caffarelli et L. Nirenberg, dans [BCN97a], prouvent que le résultat de monotonie du théorème 0.5.9 reste vrai dans $\mathbb{R}^2_+$, sans aucune hypothèse sur u pour f globalement Lipschitzienne. Précisément, ils montrent le résultat qui va suivre:

Théorème 0.5.10 (**Berestycki-Caffarelli-Nirenberg**). *Soit* $u \in C^2(\overline{\mathbb{R}^2_+})$ *une solution de* (SPE) *dans* $\mathbb{R}^2_+$*, avec* $f \in Lip([0,+\infty))$*. alors,*

$$\frac{\partial u}{\partial x_2} > 0 \quad dans \quad \mathbb{R}^2_+.$$

Dans [FS17], A. Farina et B. Sciunzi démontrent un résultat de monotonie pour les solutions éventuellement non bornées de (SPE) pour une fonction f localement Lipschitzienne. En particulier, ce résultat donne une réponse positive au problème ouvert 0.5.1 et recouvre le théorème 0.5.10 qui est vrai pour une fonction f globalement Lipschitzienne. Plus précisément, ils démontrent le théorème suivant:

Théorème 0.5.11 (Farina-Sciunzi). *Soit* $f \in Lip_{loc}([0,+\infty))$ *et* $u \in C^2(\overline{\mathbb{R}^2_+})$ *une solution de* (SPE) *dans* $\mathbb{R}^2_+$*. Alors*

$$\frac{\partial u}{\partial x_2} > 0 \quad dans \quad \mathbb{R}^2_+.$$

La preuve du théorème 0.5.11 est basée sur une variante de la méthode des hyperplans mobiles qui est appelée *méthode du plan rotatif* ("rotating plane method" en Anglais). Cette méthode consiste à comparer la solution u de (SPE) et sa symétrique par rapport à une droite passant par un point $(x_0, s) \in \mathbb{R}^2_+$ et de pente $\tan(\theta)$ avec $\theta \in (0, \frac{\pi}{2})$. En réalisant cette construction, le triangle formé par cette droite et les axes (Ox) et (Oy) est borné, le fait que nous soyons en dimension 2 jouant ici un rôle. Les auteurs peuvent ensuite utiliser le principe du maximum pour des domaines bornés et pour des domaines de petit volume (voir [FS16], [FS17]) pour compléter la procédure.

Cette construction est purement bidimensionnelle et il ne semble pas possible de la reproduire en dimensions supérieures. Le cas $N \geq 3$ reste donc un problème ouvert.

Lorsque l'on considère des domaines non bornés Ω différents de $\mathbb{R}^N_+$, très peu de résultats sont connus. À cet égard, nous citons un résultat de H. Berestycki, L.A. Caffarelli et L. Nirenberg (voir [BCN97b]) étendant le théorème 0.5.4 de M.J. Esteban et P.L. Lions (voir [EL82]) qui n'est valable que lorsque $f(0) \geq 0$. Précisément, ils prouvent le théorème suivant:

Théorème 0.5.12 (Berestycki-Caffarelli-Nirenberg). *Supposons que* $\Omega \subset \mathbb{R}^N$ *est un épigraphe régulier, localement Lipschitzien et coercif. Soit u une solution de* (SPE) *où* $f \in Lip_{loc}([0,+\infty))$*. Alors*

$$\frac{\partial u}{\partial x_N} > 0 \quad dans\ \Omega.$$

Nous étudions également le cas des épigraphes coercifs. Grâce à la méthode développée dans notre premier article (voir chapitre 1), nous prouvons le résultat suivant:

Théorème 0.5.13 (B.-Farina-Sciunzi). *Soit* Ω *un épigraphe coercif. Supposons* $f \in Lip_{loc}([0,+\infty))$ *et soit* $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *une solution distributionnelle de* (SPE)*. Alors u est strictement croissante dans la direction* x_N*, i.e.,* $\frac{\partial u}{\partial x_N} > 0$ *dans* Ω*.*

Il n'y a aucune hypothèse concernant la régularité de Ω et le signe de $f(0)$, par conséquent, nous retrouvons le théorème 0.5.4 de M.J. Esteban et P.L. Lions et le théorème 0.5.12 de H. Berestycki, L.A. Caffarelli et L. Nirenberg.

Lorsque f est une fonction continue et décroissante, nous mettons également en évidence un résultat de monotonie pour les solutions (bornées ou non) de (SPE) dans des domaines Ω qui ne sont pas nécessairement des épigraphes mais qui satisfont la propriété suivante:

$\mathcal{R}$: Ω `contient la réflection (par rapport à l'hyperplan` $\{x_N = \theta\}$`) de toutes les bandes` $\Omega \cap \{x_N < \theta\}$.

Plus précisément, nous énonçons le résultat suivant :

Théorème 0.5.14 (B.-Farina-Sciunzi). *Soit Ω un domaine non borné minoré satisfaisant la propriété $\mathcal{R}$ et soit $f : [0, \infty) \mapsto \mathbb{R}$ une fonction décroissante et localement Lipschitzienne. Soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de* (SPE) *avec une croissance sous-exponentielle sur les bandes finies, i.e., pour tout $R > 0$,*

$$\limsup_{\substack{|x|\to\infty,\\ x\in\Omega\cap\{x_N<R\}}} \frac{\ln u(x)}{|x|} \leq 0.$$

alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.

Ce résultat s'applique à des domaines non bornés très généraux (qui ne sont pas nécessairement des épigraphes), comme $\Omega_4 = \bigcup_{k\in\mathbb{Z}}\{x \in \mathbb{R}^2 : |x_1 - (3+4k)| < 1\,,\ x_2 > -1\} \bigcup \{x \in \mathbb{R}^2 : x_2 > 0\}$ (voir figure (7))(en fait, Ω_4 n'est jamais un épigraphe, c'est-à-dire qu'il n'existe pas de vecteur unitaire ν de $\mathbb{R}^2$ permettant de le représenter comme un épigraphe par rapport à ν), ou l'orthant ouvert $\Omega_5 = \{x \in \mathbb{R}^N : x_1 > 0, \ldots, x_N > 0\}$ qui vérifie la propriété $\mathcal{R}$, mais ce n'est pas un épigraphe par rapport à e_N.

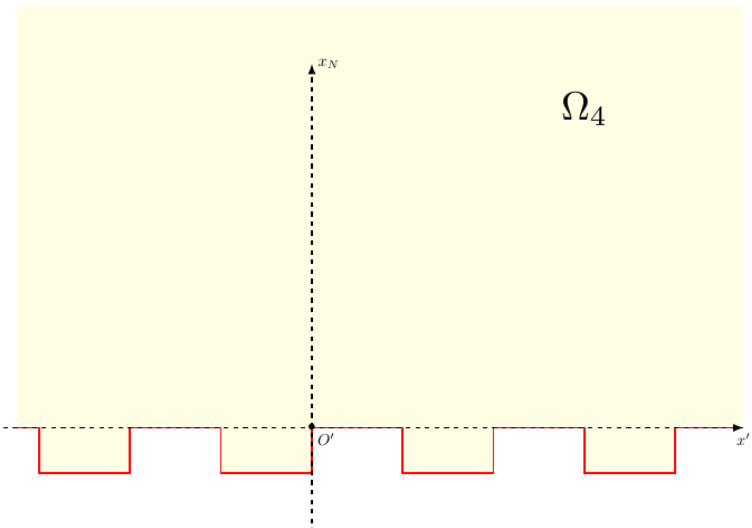

Figure 7: Ω_4

Nous voyons que le résultat ci-dessus caractérise les domaines euclidiens propres pour lesquels le problème de Dirichlet homogène (SPE) admet une solution monotone. Ceci peut être formulé de la manière équivalente suivante: le problème (SPE) avec f décroissante, admet une solution monotone sur un domaine $\Omega \subsetneq \mathbb{R}^N$, si et seulement si, Ω est contenu dans un demi-espace affine (dont la normale intérieure est notée ν) et, pour tout $x \in \Omega$, la demi-droite ouverte $\{x + t\nu,\ t > 0\}$ est contenue dans Ω. C'est-à-dire, si et seulement si,

Ω est contenu dans un demi-espace affine (dont la normale intérieure est notée ν) et Ω est ν-invariant.

Jusqu'à présent, nous avons vu qu'à moins de faire des hypothèses fortes sur le domaine Ω (voir le théorème 0.5.13) ou sur la non-linéarité f (voir le théorème 0.5.14), nous ne pouvons pas établir de résultats de monotonie indépendants du signe de $f(0)$. Cependant, lorsque nous considérons des solutions éventuellement non bornées du problème (SOP), nous pouvons prouver un résultat de monotonie pour une large classe d'épigraphes minorés, quel que soit le signe de $f(0)$. C'est le contenu du théorème 0.5.15 ci-dessous. Pour cela, nous introduisons la classe de fonctions suivante:

Définition 0.5.3. Soient $N \geq 2$ et $\gamma \in (0,1]$. On désigne par $\mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ l'ensemble des fonctions $h \in C^1(\mathbb{R}^{N-1})$ tel que ∇h est borné et globalement γ-Hölderien, i.e., tel que:

$$\begin{aligned}\|\nabla h\|_{C^{0,\gamma}(\mathbb{R}^{N-1})} &:= \|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})} + [\nabla h]_{C^{0,\gamma}(\mathbb{R}^{N-1})} \\ &:= \sup_{x\in\mathbb{R}^{N-1}} |\nabla h(x)| + \sup_{x,y\in\mathbb{R}^{N-1}, x\neq y} \frac{|\nabla h(x) - \nabla h(y)|}{|x-y|^\gamma} < +\infty.\end{aligned}$$

Pour les épigraphes définis par ce type de fonctions, nous prouvons le résultat suivant:

Théorème 0.5.15 (B.-Farina-Sciunzi). *Supposons que $N \geq 2$, $\gamma \in (0,1]$. Soit Ω un épigraphe minoré et définit par une fonction $g \in \mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ et soit $f \in Lip_{loc}(\mathbb{R})$.*
Soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de (SOP) *avec $\mathfrak{c} \neq 0$ et tel que ∇u est borné sur les bandes finies. Alors, u est strictement croissante dans la direction x_N, i.e.*

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega.$$

En particulier, le théorème ci-dessus s'applique aux solutions bornées de (SOP). En effet, les solutions bornées de (SOP) ont également un gradient borné (voir le corollaire 2.5.2).

Lorsque $f(0) \geq 0$, le théorème 0.5.15 est une application du théorème 0.5.8. En effet, si $g \in \mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ alors g est globalement Lipschitzienne, et comme ∇u est borné sur des bandes finies, u est borné sur les bandes finies. Le théorème 0.5.8 s'applique donc.

0.5.3 Extensions aux épigraphes de classe $\mathcal{G}$ et minorés

Nous avons noté que nous pouvions étendre nos résultats de monotonie à une classe plus large d'épigraphes minorés. En effet, lors de nos preuves, nous sommes amenés à considérer les suites de translatées $g_k(\cdot) = g(\cdot + (x^k)')$ et en étudier la convergence. C'est la raison pour laquelle, nous définissons une nouvelle classe de fonctions où la convergence d'une telle suite est directement supposée.

Definition 0.5.1. Soit $N \geq 2$. On dit qu'une fonction continue $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ appartient à la classe $\mathcal{G}$, si elle satisfait la propriété de compacité suivante:

$(\mathcal{P})$ *Toute suite (g_k) de translatées de g, qui est bornée en un point fixe de $\mathbb{R}^{N-1}$, admet une sous-suite convergeant uniformément sur tout compact de $\mathbb{R}^{N-1}$.*

Avant d'énoncer les nouveaux résultats de monotonie, montrons à quel point la classe $\mathcal{G}$ est grande. Ci-après, nous donnons une liste longue (mais non exhaustive) des membres de $\mathcal{G}$

1. Les fonctions uniformément continues sur $\mathbb{R}^{N-1}$ appartiennent à $\mathcal{G}$.

2. Les fonctions coercives sur $\mathbb{R}^{N-1}$ appartiennent à $\mathcal{G}$.

   En particulier, toute fonction continue sur $\mathbb{R}^{N-1}$ tel que $\lim\limits_{|x|\mapsto\infty} g(x) \in (-\infty, +\infty]$ appartient à $\mathcal{G}$.

3. On désigne par $\mathsf{G}(\mathbb{R}^{N-1})$ l'ensemble des fonctions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ satisfaisant la propriété suivante: *Il existe une* **bijection continue** *$\phi : \mathbb{R} \to \mathbb{R}$ tel que $\phi \circ g \in \mathcal{G}$.*
   Il est immédiat de vérifier que $\mathsf{G}(\mathbb{R}^{N-1}) \subset \mathcal{G}$. De plus, la famille $\mathsf{G}(\mathbb{R}^{N-1})$ contient strictement celle des *fonctions uniformément continues* sur $\mathbb{R}^{N-1}$ et celle des *fonctions continues et coercives* sur $\mathbb{R}^{N-1}$, comme illustrer par les exemples suivant:

   (3a) Par exemple, les fonctions $g_1(x_1) = e^{x_1}$ si $N = 2$, et $g(x) = e^{x_1+\sum_{j=2}^{N-1}\cos^j(x_j)}$ si $N \geq 3$, ne sont ni uniformément continues, ni coercives sur $\mathbb{R}^{N-1}$. Pourtant, elles appartiennent à la classe $\mathsf{G}(\mathbb{R}^{N-1})$ (on peut prendre par exemple, la fonction:
   $$\phi(t) = \begin{cases} t & \text{si} \quad t \leq 0, \\ \log(t+1) & \text{si} \quad t > 0, \end{cases}$$
   dans la définition précédente et observer que $\phi \circ g \in \mathcal{G}$ est uniformément continue, donc $\phi \circ g \in \mathcal{G}$). Le même argument montre également que les fonctions $g(x_1) = e^{e^{x_1}}$ et $g(x) = e^{e^{x_1+\sum_{j=2}^{N-1}\cos^j(x_j)}}$ appartiennent à $\mathsf{G}(\mathbb{R}^{N-1})$. Puisque cet argument peut être itérer, on voit que la classe $\mathsf{G}(\mathbb{R}^{N-1})$ contient des fonctions régulières, minorées, qui ne sont jamais uniformément continues ou coercives, et avec une croissance large à l'infini. Notons également que la fonction g_1 étant convexe, son épigraphe satisfait la condition du cône extérieur uniforme.

   (3b) Supposons $N \geq 2$ et soit $g \in C^2(\mathbb{R}^{N-1})$ une fonction strictement positive telle que $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$, alors la fonction $\sqrt{g}$ est globalement Lipschitzienne sur $\mathbb{R}^{N-1}$ (voir le lemme I dans [Gla63] pour $N = 2$). Ainsi, on a $g \in \mathsf{G}(\mathbb{R}^{N-1})$. Plus généralement, toute fonction $g \in C^2(\mathbb{R}^{N-1})$, minorée et tel que $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$ est un membre de la classe $\mathsf{G}(\mathbb{R}^{N-1})$.
   Par exemple, la fonction $g = g(x_1, \ldots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1}\sin(jx_j)$ appartient à $\mathsf{G}(\mathbb{R}^{N-1})$ pour tout $N \geq 3$.

4. Pour $N = 2$, toute fonction continue $g : \mathbb{R} \to \mathbb{R}$ tel que $\ell_- := \lim\limits_{x\mapsto-\infty} g(x) \in (-\infty, +\infty]$ et $\ell_+ := \lim\limits_{x\mapsto+\infty} g(x) \in (-\infty, +\infty]$ appartient à $\mathcal{G}$. Ainsi, toute fonction continue quasi-convexe (resp. quasi-concave) minorée appartient à la classe $\mathcal{G}$. En particulier,

les fonctions continues monotones et minorées ainsi que les fonctions convexes et minorées appartiennent à $\mathcal{G}$.

5. Supposons $2 \leq n < N$ et soit $g : \mathbb{R}^{n-1} \to \mathbb{R}$ un élément de $\mathcal{G}$. Alors, la fonction $\tilde{g} : \mathbb{R}^{N-1} \to \mathbb{R}$ définie par $\tilde{g}(x_1, \ldots, x_{N-1}) = g(x_1, \ldots, x_{n-1})$ satisfait la propriété de compacité $(\mathcal{P})$, comme étant une fonction définie sur $\mathbb{R}^{N-1}$. Ainsi, $\tilde{g} \in \mathcal{G}$, comme étant une fonction définie sur $\mathbb{R}^{N-1}$.

   En particulier, pour $N \geq 2$, les fonctions $g(x_1, \ldots, x_N) = e^{e^{x_1}}$ et $g(x_1, \ldots, x_N) = e^{x_1} - 4\arctan(x_1) - 2$ appartiennent à $\mathcal{G}$, et sont minorées et leurs épigraphes satisfont la condition du cône extérieur uniforme.

6. Supposons $N \geq 2$. Soient $g \in \mathcal{G}$ et $T : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ une transformation de la forme $T(x) = Ax + b$, où A est une matrice réelle inversible et $b \in \mathbb{R}^{N-1}$. Alors, $g \circ T \in \mathcal{G}$. En particulier, ce résultat s'applique quand T est une isométrie de $\mathbb{R}^{N-1}$.

7. Supposons $N \geq 2$ et $\lambda \geq 0$. Soit $g, \tilde{g}$ deux fonctions minorées appartenant à $\mathcal{G}$. Alors, $\lambda g \in \mathcal{G}$ et $g + \tilde{g} \in \mathcal{G}$.

   En particulier, pour $N \geq 2$, la fonction $g(x_1, \ldots, x_N) = x_1^4 + e^{x_2}$ appartient à $\mathcal{G}$, elle est minorée et son épigraphe satisfait la condition du cône extérieur uniforme.

8. En combinant les items (1)-(7), on peut facilement construire des exemples de fonctions g appartenant à $\mathcal{G}$ pour toute dimension $N \geq 2$. En particulier, on peut construire des éléments de $\mathcal{G}$ minorés, qui ne sont jamais uniformément continus ni coercifs, avec des croissances larges à l'infini et dont les épigraphes satisfont la condition du cône extérieur uniforme.

Au vu des discussions ci-dessus, nous pouvons maintenant énoncer les résultats de monotonie pour des épigraphes continus et minorés définis par des fonctions g appartenant à la classe $\mathcal{G}$. Commençons par l'extension du théorème 0.5.6 et du corollaire 0.5.1.

Théorème 0.5.16 (B-Farina-Sciunzi). *Soient $N \geq 2$ et Ω un épigraphe minoré et défini par une fonction $g \in \mathcal{G}$. Supposons $f \in Lip_{loc}([0, +\infty))$ avec:*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0. \tag{0.5.2}$$

(i) Si $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ est une solution distributionnelle de (SPE) *qui est bornée et uniformément continue sur les bandes finies.*
Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.
Si f est globalement Lipschitzienne, la même conclusion reste vraie sous la seule hypothèse que u est uniformément continue sur les bandes finies.

(ii) Si $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ est une solution classique de (SPE) *tel que $\nabla u \in L^\infty(\Omega)$. Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*

Si on suppose, en plus, que l'épigraphe satisfait la condition du cône extérieur uniforme, alors on peut également étendre les résultats 0.5.7 et 0.5.8 aux épigraphes définis par des fonctions $g \in \mathcal{G}$ minorées.

Théorème 0.5.17 (B.-Farina-Sciunzi). *Soient $N \geq 2$ et Ω un épigraphe défini par une fonction $g \in \mathcal{G}$. Supposons également que Ω est minoré et satisfait la condition du cône extérieur uniforme.*

(i) Supposons $f \in Lip([0, +\infty))$ avec $f(0) \geq 0$ et soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de (SPE) *avec une croissance au plus exponentielle sur les bandes finies. Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*

(ii) Supposons $f \in Lip_{loc}([0, +\infty))$ avec $f(0) \geq 0$ et soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle de (SPE) *qui est bornée sur les bandes finies. Alors u est strictement croissante dans la direction x_N, i.e., $\frac{\partial u}{\partial x_N} > 0$ dans Ω.*

Notons que les théorèmes 0.5.16 et 0.5.17 s'appliquent aux exemples de fonctions décrits dans les items précédents (1)-(7). Ils recouvrent également le résultat parut récemment dans le preprint [GMS24] où les auteurs prouvent la stricte monotonie des solutions classiques de (SPE) avec $f(t) = t^q$ et $q \geq 1$.[19]

Nous voudrions maintenant faire remarquer que tout les résultats énoncés précédemment le sont pour des fonctions f globalement ou localement Lipschitzienne, à l'exception du théorème 0.5.14. Cette hypothèse est optimale, comme le montre l'exemple suivant:

Exemple 0.5.1. Supposons $N \geq 2$ et soit Ω le demi-espace $\{x \in \mathbb{R}^N \, : \, x_N > 0\}$. La fonction bornée

$$u(x) = \begin{cases} 1 - (x_N - 1)^4 & \text{si} \quad 0 \leq x_N \leq 1, \\ 1 & \text{si} \quad x_N > 1, \end{cases} \tag{0.5.3}$$

est une solution classique de (SPE), où f correspond à la fonction décroissante suivante:

$$f(t) = \begin{cases} 12 & \text{si} \quad t < 0, \\ 12\sqrt{1-t} & \text{si} \quad 0 \leq t \leq 1, \\ 0 & \text{si} \quad t > 1. \end{cases} \tag{0.5.4}$$

La fonction f est globalement Hölderienne, mais n'est pas localement Lipschitzienne sur $\mathbb{R}$ et $\frac{\partial u}{\partial x_N} = 0$ sur $\{(x', x_N) \in \mathbb{R}^N, \, x_N > 1\}$.

0.5.4 Quelques applications sur la classification et les résultats de non-existence

Dans cette sous-section, on présente les premières applications des résultats de monotonie décrits dans la section précédente. Elles concernent la classification des solutions du problème suivant:

[19] Il est évident que toute fonction semi-coercive et continue est minorée et appartient à notre classe $\mathcal{G}$ (voir les points (5) et (6) ci-dessus). Cette observation et le lemme A.2 de [GMS24] montrent également que le cas de l'épigraphe convexe est couvert par les techniques développées dans notre premier article.

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \Omega, \\ \quad u \geq 0 & \text{dans} \quad \Omega, \\ \quad u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{SPE+}$$

où, Ω est un épigraphe minoré et f une fonction localement Lipschitzienne.

Le premier résultat que l'on cite est un résultat de type Liouville dû à M.J. Esteban et P.L. Lions (voir [EL82]), qui considèrent les solutions de (SPE+) dans un épigraphe régulier et coercif et prouvent:

Théorème 0.5.18 (Esteban-Lions). *Soit Ω un épigraphe coercif régulier et $f \in Lip_{loc}([0, +\infty))$ tel que $f(0) \geq 0$. Supposons qu'il existe une solution $u \in C^2(\overline{\Omega})$ de (SPE+) satisfaisant:*

$$\lim_{\substack{x\in\Omega, \\ |x|\longrightarrow\infty}} u(x) = 0.$$

Alors $u \equiv 0$.

Plus tard, A. Farina, dans [Far07], fournit une classification des solutions de l'équation de Lane-Emden $-\Delta u = u^p$ dans des épigraphes coercifs et pour toutes dimensions. Plus précisément, il démontre le résultat suivant:

Théorème 0.5.19 (Farina). *Soient $0 < \alpha < 1$ et Ω un épigraphe coercif de classe $C^{2,\alpha}$. Soit $u \in C^2(\overline{\Omega})$ une solution de*

$$\begin{cases} -\Delta u = u^p & \textit{dans} \quad \Omega, \\ \quad u \geq 0 & \textit{dans} \quad \Omega, \\ \quad u = 0 & \textit{sur} \quad \partial\Omega. \end{cases} \tag{0.5.5}$$

avec

$$\begin{cases} 1 < p < +\infty & \textit{si } N \leq 10, \\ 1 < p < p_{JL}(N) := \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \textit{si } N \geq 11. \end{cases}$$

Alors,

$$u \equiv 0 \quad \textit{dans } \Omega.$$

Si, on suppose en plus que u est une solution ***bornée*** *de (0.5.5) et*

$$\begin{cases} 1 < p < +\infty & \textit{si } N \leq 11, \\ 1 < p < p_{JL}(N-1) & \textit{si } N \geq 12. \end{cases}$$

Alors,

$$u \equiv 0.$$

L'exposant p_{JL} est appelé exposant de stabilité de Joseph-Lundgren.

Ce théorème est le premier résultat de classication des solutions éventuellement non-bornées de l'équation de Lane-Emden dans des épigraphes coercifs réguliers.

Dans [DF22], L. Dupaigne et A. Farina considèrent les solutions bornées de (SPE+) dans des épigraphes seulement localement Lipschitzien et avec une fonction f strictement positive de classe C^1. Dans ce cadre, les auteurs démontrent le résultat suivant:

Théorème 0.5.20 (Dupaigne-Farina). *Soient $\Omega \subset \mathbb{R}^N$ un épigraphe coercif localement Lipschitzien et $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ une solution bornée de* (SPE+).
Supposons que $f \in C^1([0,+\infty))$, $f(t) > 0$ pour $t > 0$ et $2 \leq N \leq 11$. Alors $f(0) = 0$ et $u \equiv 0$.

Ce résultat s'applique à $f(t) = t^p$, ainsi il recouvre le second point du théorème 0.5.19 en dimension $N \leq 11$.

Contrairement au théorème 0.5.18 de M.J Esteban et P.L Lions, dans le théorème 0.5.20, il n'y a pas d'hypothèse sur la régularité de l'épigraphe Ω. Cependant, le théorème 0.5.20 s'applique à des fonctions f de classe C^1 et est limité par la dimension N comprise entre 2 et 11.

Pour prouver ce théorème, les auteurs utilisent le caractère borné et la monotonie des solutions de (SPE+) (qui provient du théorème 0.5.12) afin d'assurer que la fonction

$$v(x_1, \cdots x_{N-1}) = \lim_{x_N \to +\infty} u(x', x_N),$$

existe, et qu'elle est une solution classique et stable[20] de l'équation $-\Delta v = f(v)$ dans $\mathbb{R}^{N-1}$. La conclusion du théorème 0.5.20 est une application du théorème de type Liouville de A. Farina et L. Dupaigne, qui est valable dans l'espace entier jusqu'à la dimension 10 (voir le théorème 1 dans [DF22]).

Dans notre premier article, (voir le chapitre 1), grâce à nos résultats de monotonie dans des épigraphes définis par des fonctions appartenant à la classe $\mathcal{G}$ (voir 2.2.2 dans la sous-section 0.5.3), nous avons pu utiliser la technique décrite ci-dessus et mettre en évidence des résultats de type Liouville qui recouvrent et complètent ceux énoncés précédemment.

Commençons par un résultat de classification des solutions bornées de (SPE+) en dimensions $2 \leq N \leq 11$:

Théorème 0.5.21 (B.-Farina-Sciunzi). *Soit Ω un épigraphe minoré definit par une fonction $g \in \mathcal{G}$ et satisfaisant la condition du cône extérieur uniforme.*
Soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle et bornée de (SPE+) *Supposons que $f \in C^1([0,+\infty))$, $f(t) > 0$ pour $t > 0$ et $2 \leq N \leq 11$, alors $u \equiv 0$ et $f(0) = 0$.*

Le théorème précédent reste vrai pour $N \geq 12$, si on ajoute une hypothèse sur le comportement à l'origine de f.

[20] i.e, pour tout $\phi \in C^1_c(\mathbb{R}^{N-1})$, on a

$$\int_{\mathbb{R}^{N-1}} f'(v)\phi^2 \leq \int_{\mathbb{R}^{N-1}} |\nabla \phi|^2.$$

Théorème 0.5.22 (**B.-Farina-Sciunzi**). *Supposons $N \geq 12$ et soit Ω un épigraphe minoré définit par une fonction $g \in \mathcal{G}$ et satisfaisant la condition du cône extérieur uniforme.*
Soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle et bornée de (SPE+) *ou $f \in C^1([0,+\infty))$ satisfait $f(t) > 0$ pour $t > 0$ et $\liminf_{t\to 0^+} \frac{f(t)}{t^s} > 0$, pour un réel $s \in \left[0, \frac{N-3}{N-5}\right)$. alors $u \equiv 0$ et $f(0) = 0$.*

Notons que le théorème 0.5.21 et le théorème 0.5.22 impliquent immédiatement le résultat de non-existence lorsque $f(0) > 0$ suivant:

Corollaire 0.5.4 (**B.-Farina-Sciunzi**). *Soient $N \geq 2$ et $\Omega \subset \mathbb{R}^N$ un épigraphe minoré, uniformément continu et satisfaisant la condition du cône extérieur uniforme.*
Si $f \in C^1([0,+\infty))$ satisfait $f(t) > 0$ pour $t \geq 0$, alors le problème (SPE+) *ne possède pas de solution distributionnelle bornée de classe $C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$.*

Le prochain résultat concerne la classe naturelle de non-linéarités introduite dans [DF10]:

$$\begin{cases} f \in C^1([0,+\infty)) \cap C^2((0,+\infty)), \quad f(0) = 0, \\ \\ f > 0, \ \textit{croissante et convexe dans } (0,+\infty) \\ \\ \qquad s.t. \qquad \lim\limits_{u\to 0^+} \frac{f'(u)^2}{f(u)f''(u)} := q_0 \in [0,+\infty] \end{cases} \tag{0.5.6}$$

où l'on pose $\frac{f'(u)^2}{f(u)f''(u)} = +\infty$ si $f''(u) = 0$.

Notons que l'on a nécessairement $q_0 \in [1,+\infty]$ (voir le lemme 1.4 dans [DF10]).

Le représentant typique de cette classe est donné par la fonction $f(u) = u^p$, $p > 1$. Dans ce cas, $\frac{f'(u)^2}{f(u)f''(u)} = \frac{p}{p-1}$ et donc q_0 coïncide avec l'exposant conjugué de p. Par conséquent, si on définit $p_0 \in [1,+\infty]$ comme l'exposant conjugué de q_0 par $\frac{1}{p_0} + \frac{1}{q_0} = 1$, on a que l'exposant p_0 peut être vu comme la "mesure" de la planéité de f à l'origine. Les fonctions $f_n(u) = e^u - \sum_{k=0}^n \frac{u^k}{k!}$ sont également des éléments de la classe définie ci-dessus. Il est facile de voir que $q_0(f_n) = \frac{n+1}{n}$ et donc $p_0(f_n) = n+1$.

Pour cette classe de non-linéarités, nous avons montrer le résultat suivant:

Théorème 0.5.23 (**B.-Farina-Sciunzi**). *Supposons $N \geq 12$ et soit Ω un épigraphe minoré définit par une fonction $g \in \mathcal{G}$ et satisfaisant la condition du cône extérieur uniforme.*
Soit $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ une solution distributionnelle et bornée de (SPE+) *où f satisfait* (0.5.6).
Supposons que p_0, l'exposant conjugué de q_0, satisfait:

$$1 \leq p_0 < p_{JL}(N-1), \tag{0.5.7}$$

où p_{JL} est l'exposant de stabilité de Joseph-Lundgren définit par

$$p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

Alors $u \equiv 0$.

Notons que le théorème précédent s'applique à $f(u) = u^p$ si $1 \leq p < p_{JL}(N-1)$ et à f_n si $n+1 < p_{JL}(N-1)$.

Les preuves des théorèmes 0.5.22 et 0.5.23 sont basées sur la méthode décrite juste après le théorème 0.5.20 et sur l'utilisation des résultats de type Liouville pour des solutions stables établis dans [Far07], [DF10] et [Far15].

Le résultat suivant est une conséquence immédiate du théorème 0.5.13.

Corollaire 0.5.5 (**B.-Farina-Sciunzi**). *Lorsque l'épigraphe* Ω *est coercif, la conclusion des théorèmes 0.5.21-0.5.23 et du corollaire 0.5.4 reste vraie sous la seule hypothèse de continuité de* g. *Nous n'avons pas besoin d'exiger la condition de cône extérieur uniforme pour l'épigraphe.*

On conclut cette sous-section avec un résultat de classification pour des solutions de (SPE+) qui tendent vers zéro à l'infini.

Théorème 0.5.24 (**B.-Farina-Sciunzi**). *Soient* $N \geq 2$ *et* Ω *un épigraphe définit par une fonction* $g \in \mathcal{G}$. *Supposons également que* Ω *est minoré et satisfait la condition du cône extérieur. Supposons* $f \in Lip_{loc}([0,+\infty))$ *et soit* $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *une solution distributionelle* (SPE+) *tel que:*

$$\lim_{\substack{x\in\Omega,\\ |x|\longrightarrow\infty}} u(x) = 0. \tag{0.5.8}$$

alors $u \equiv 0$ *et* $f(0) = 0$.
Lorsque l'épigraphe Ω *est coercif, la conclusion ci-dessus est vraie sous la seule hypothèse de continuité de* g.

Notons que dans le résultat précédent, on ne fait aucune hypothèse que f (autre que $f \in Lip_{loc}([0,+\infty))$) et sur le caractère bornée de u.

0.6 Nouveaux principes de comparaison dans des ensembles non-bornés

Nos nouveaux résultats de monotonie décrits dans la section précédente reposent sur la méthode des hyperplans mobiles, adaptée à la géométrie de l'épigraphe. Expliquons cela plus en détail.

Si on considère une solution u du problème (SPE), la méthode des hyperplans mobiles consiste à montrer l'égalité suivante:

$$\Lambda := \{t > 0 \ : \ u \leqslant u_\theta \ \text{ dans } \ \Sigma_\theta\,, \ \forall\, 0 < \theta < t\} = (0,+\infty). \tag{0.6.1}$$

où $\Sigma_\theta = \{x = (x', x_N) \in \Omega, x_N < \theta\}$ et $u_\theta(x) = u(x', 2\theta - x_N)$.

Cela signifie que, pour tout Σ_t, $t > 0$, la solution u se trouve sous u_t, sa réflexion par rapport à l'hyperplan $\{x_N = t\}$.

On peut voir que si l'égalité (0.6.1) est vraie alors u est croissante dans la direction x_N. On obtient la stricte croissance en utilisant le lemme de Hopf.

Pour démontrer l'égalité (0.6.1), il faut d'abord prouver que Λ n'est pas vide. Lorsque Σ_θ est borné, c'est une conséquence d'une application appropriée du principe du maximum classique. Dans notre cas, Σ_θ pourrait être non borné (éventuellement non connexe et de mesure infinie) et a priori u n'est pas bornée.

Pour surmonter ces difficultés supplémentaires dans notre premier article (voir chapitre 1), nous avons prouvé quelques nouveaux principes de comparaison pour une grande classe de sous-ensembles euclidiens non bornés, que nous appelons *ensembles à bonnes sections* (voir sous-section 0.6.1 ci-dessous). En particulier, ces résultats généralisent ceux déjà existants dans le cas particulier des bandes finies, que nous rappelons ci-dessous.

Dans [Far20], A. Farina développe des principes de comparaison pour des fonctions définies sur des bandes de la forme $\mathbb{R}^{N-1} \times [a, b]$. En effet, il montre le théorème suivant:

Théorème 0.6.1 (Farina).

1) Soit $N \geq 2$, $M > 0$ et supposons $f \in Lip_{loc}([0, +\infty))$. Alors il existe $\varepsilon = \varepsilon(f, M) > 0$ tel que, pour tout $(a, b) \subset \mathbb{R}$ avec $0 < b-a < \varepsilon$ et tout $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$ satisfaisant:

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b), \\ |u|, |v| \leq M & \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b), \\ u \leq v & \textit{sur} \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

on a,

$$u \leq v \quad \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b).$$

2) Soit $N \geq 2$ et supposons $f \in Lip([0, +\infty))$. Alors il existe $\varepsilon = \varepsilon(f) > 0$ tel que, pour tout $(a, b) \subset \mathbb{R}$ avec $0 < b-a < \varepsilon$ et tout $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$, avec, au plus, une croissance polynomiale à l'infini et satisfaisant:

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b) \\ u \leq v & \textit{sur} \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

on a

$$u \leq v \quad \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b).$$

3) Soit $N \geq 2$ et supposons $f \in C^0([0, +\infty))$ une fonction décroissante. Alors, pour tout $(a, b) \subset \mathbb{R}$ et tout $u, v \in C^2(\mathbb{R}^{N-1} \times [a, b])$, avec, au plus, une croissance polynomiale à l'infini et satisfaisant:

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b) \\ u \leq v & \textit{sur} \quad \partial(\mathbb{R}^{N-1} \times (a, b)), \end{cases}$$

on a

$$u \leq v \quad \textit{dans} \quad \mathbb{R}^{N-1} \times (a, b).$$

Dans ce contexte, H. Berestycki, L.A. Caffarelli et L. Nirenberg, dans [BCN96], décrivent un principe du maximum pour un `opérateur linéaire` et pour des fonctions définies sur des bandes ayant une croissance exponentielle:

Théorème 0.6.2 (**Berestycki-Caffarelli-Nirenberg**). *Dans* $\Omega = \mathbb{R}^{N-1} \times (a,b)$, *supposons que* $w \in W^{2,\alpha}_{loc}(\Omega) \cap C(\overline{\Omega})$ *est une fonction satisfaisant*

$$\begin{cases} -\Delta w + c(x,y)w \leq 0 & \textit{dans} \quad \Omega, \quad \textit{avec } |c| \leq \gamma, \\ \quad w \leq 0 & \textit{dans} \quad \partial\Omega, \end{cases}$$

et

$$w \leq Ce^{\mu |x|},$$

où γ, C, *et* μ *sont des constantes positives. Il existe une constante* δ, *dépendant seulement de* N, γ *et* μ, *tel que si*

$$0 < b - a < \delta,$$

alors

$$w \leq 0 \ \textit{dans}\ \Omega.$$

0.6.1 Ensembles avec une bonne section

Dans cette sous-section, nous présentons les ensembles sur lesquels nous mettons en évidence nos nouveaux principes de comparaison:

Définition 0.6.1 (**Ensembles avec une "bonne section" dans une direction**). Soit $N \geq 2$ et soient Ω un ouvert de $\mathbb{R}^N$ et ν un vecteur unitaire de $\mathbb{R}^N$. On note $\mathbb{R}\nu$ l'espace vectoriel généré par le vecteur ν et par $\{\nu\}^\perp$ l'espace vectoriel orthogonal de $\mathbb{R}\nu$ dans $\mathbb{R}^N$.

(i) On dira que Ω est `localement borné dans la direction` ν si

$$\forall R > 0 \quad C_\nu(R) = (B'(0', R) \times \mathbb{R}\nu) \cap \Omega \quad \text{ est un sous ensemble borné de } \mathbb{R}^N. \tag{0.6.2}$$

Ici $B'(0', R)$ désigne la boule ouverte de $\mathbb{R}^{N-1}$ de rayon R centrée à l'origine $0'$ de $\{\nu\}^\perp$.

(ii) Pour tout $x' \in \{\nu\}^\perp$, on définit l'ensemble $S^\nu_{x'} := (\{x'\} \times \mathbb{R}\nu) \cap \Omega$.

On dira que Ω a une `section bornée dans la direction` ν si

$$\mathtt{S}_\nu(\Omega) := \sup_{x' \in \{\nu\}^\perp} \mathcal{L}^1(S^\nu_{x'}) < +\infty, \tag{0.6.3}$$

où $\mathcal{L}^1$ désigne la mesure de Lebesgue de $\mathbb{R}$.
Dans la suite, le réel positif $\mathtt{S}_\nu(\Omega)$ sera appelé `section de` Ω `dans la direction` ν.

(iii) On dira que Ω a une `bonne section dans la direction` ν s'il est localement borné dans la direction ν et aussi s'il possède une section bornée dans la direction ν (c'est-à-dire, s'il satisfait les deux conditions (0.6.2) et (0.6.3)).

Pour des raisons de simplicité et comme le problème (SPE) est invariant par rotation, on considérera dans la suite, que les ensembles ont une bonne section dans la direction e_N.

La classe d'ensembles définie ci-dessus est très large. En effet, tout ouvert contenu dans une bande finie possède clairement une bonne section, mais on peut aussi trouver des ouverts plus sophistiqués non contenus dans des bandes, comme $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : |x| - h(x) < y < |x| + h(x)\} \cup \{(x,y) \in \mathbb{R}^2 : -|x| - h(x) < y < -|x| + h(x)\} \cup \{(x,y) \in \mathbb{R}^2, |y| < 1\}$, où $h(x) = sinh^{-1}(e^{-|x|})$, (voir la figure 8). Cet ensemble n'est contenu dans aucun demi-espace affine, ni dans aucune bande. De plus, sa mesure de Lebesgue est infinie.

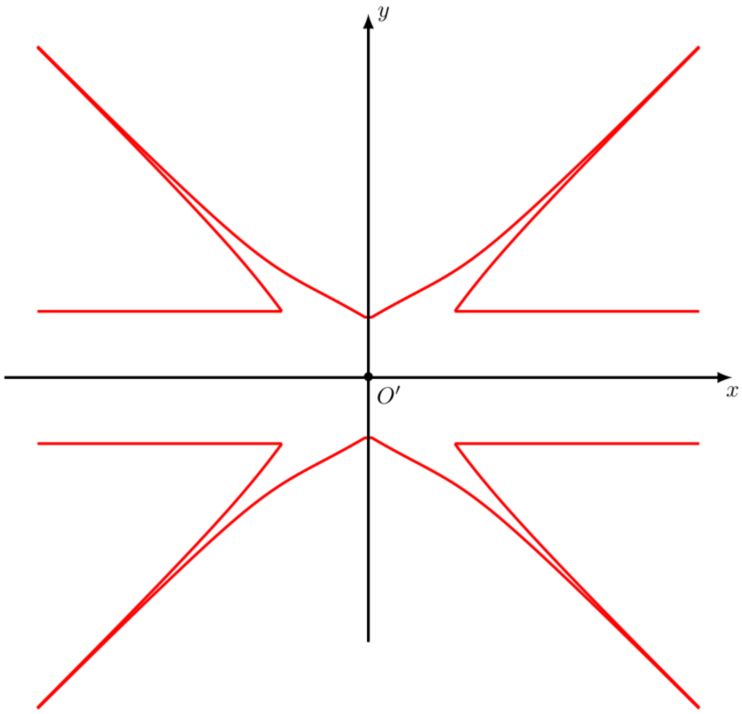

Figure 8: Ω_1

Notons que les conditions (i) et (ii) ne sont pas équivalentes. En effet , si on considère $\Omega_2 = \bigcup_{n\geq 1} \left\{(x,y) \in \mathbb{R}^2 : n < y < n + \frac{1}{2^n}\right\}$ et $\Omega_3 = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^2\}$, alors , on peut prouver que Ω_2 a une section bornée dans la direction e_2 ($\mathtt{S}_{e_2}(\Omega_2) = 1$) mais n'est pas borné dans la direction e_2. Inversement, l'ensemble Ω_3 est borné dans la direction e_2 mais sa section n'est jamais bornée puisque $S_x^{e_2} = x^2$, pour tout $x \in \mathbb{R}$.

0.6.2 Nouveaux principes de comparaison

Dans notre premier article (voir chapitre 1), nous mettons en évidence des nouveaux principes de comparaison pour des fonctions définies sur des ensembles avec des bonnes sections. C'est le contenu des résultats suivant:

Théorème 0.6.3 (B.-Farina-Sciunzi). *Soit $N \geq 2$. et Ω un ouvert $\mathbb{R}^N$ et soit ν un vecteur unitaire de $\mathbb{R}^N$.*

(i) Supposons que Ω *a une bonne section dans la direction* ν*. Soit* $f \in Lip_{loc}(\mathbb{R})$*,* $M > 0$ *et* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *satisfaisant:*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & dans \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & dans \quad \Omega, \\ u \leq v & sur \quad \partial\Omega. \end{cases} \tag{0.6.4}$$

Alors, il existe $\varepsilon = \varepsilon(f, M) > 0$ *tel que*

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \ dans\ \Omega. \tag{0.6.5}$$

(ii) Supposons que Ω *a une bonne section dans la direction* ν*. Soit* $f \in C^0(\mathbb{R})$ *une fonction décroissante,* $M > 0$ *et* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *satisfaisant:*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & dans \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & dans \quad \Omega, \\ u \leq v & sur \quad \partial\Omega. \end{cases} \tag{0.6.6}$$

Alors, $u \leq v$ *in* Ω*.*

(iii) Supposons que Ω *a une section bornée dans la direction* ν*. Soit* $f \in Lip_{loc}(\mathbb{R})$*,* $M > 0$ *et* $u, v \in Lip_{loc}(\overline{\Omega})$ *satisfaisant:*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & dans & \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & dans & \Omega, \\ |\nabla u|, |\nabla v| \leq M & p.p.\ dans & \Omega, \\ u \leq v & sur & \partial\Omega. \end{cases} \tag{0.6.7}$$

Alors, il existe $\varepsilon = \varepsilon(f, M) > 0$ *tel que*

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \ dans\ \Omega. \tag{0.6.8}$$

Ce résultat recouvre le résultat de comparaison 0.6.1 de A. Farina dans [Far20]. Cependant, pour démontrer nos résultats de monotonie, nous devons comparer des fonctions qui ont des croissances au plus exponentielle. C'est la raison pour laquelle, nous prouvons également le résultat suivant:

Théorème 0.6.4 (B.-Farina-Sciunzi). *Supposons que* $\gamma \geq 0$*,* $\delta \geq 0$*,* $N \geq 2$ *et soit* Ω *un ouvert de* $\mathbb{R}^N$ *avec une bonne section dans la direction* e_N*, le dernier vecteur de la base canonique de* $\mathbb{R}^N$*, tel que:*

$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma |x_N|} dx_N \right) < +\infty. \tag{0.6.9}$$

Soit $f = f_1 + f_2$*, avec* $f_1 \in Lip(\mathbb{R})$ *et* $f_2 : \mathbb{R} \mapsto \mathbb{R}$ *une fonction décroissante.*

Soit $a > 0$ *et* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *tel que:*

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & dans \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^\delta e^{\gamma|x|} & dans \quad \Omega, \\ u \leq v & sur \quad \partial\Omega. \end{cases}$$

Alors, il existe $\varepsilon = \varepsilon(L_{f_1}, \gamma) > 0$ *tel que:*

$$\mathtt{S}_{e_N}(\Omega) < \varepsilon \implies u \le v \text{ dans } \Omega.$$

Ici, L_{f_1} *correspond à la constante de Lipschitz de* f_1*.*

Les remarques suivantes s'imposent:

(i) Ces théorèmes recouvrent et complètent les théorèmes 0.6.1 et 0.6.2.

(ii) Tout ensemble Ω inclus dans une bande $\{x = (x', x_N) \in \mathbb{R}^N : \alpha < x_N < \beta\}$, avec $\alpha, \beta \in \mathbb{R}$, satisfait clairement l'hypothèse (0.6.9) pour tout $\gamma, \delta \ge 0$.

(iii) Notons que le théorème précédent s'applique à l'ensemble Ω_1 (voir la figure (8)). En effet, Ω_1 satisfait l'hypothèse (0.6.9) pour tout $\delta \ge 0$ et tout $\gamma \in [0, 1/2]$.

L'hypothèse concernant la taille de la section de Ω est nécessaire pour la validité du théorème 0.6.3 (item (i) and item (iii)) et du théorème 0.6.4. En effet, les fonctions $u = \sin(y)$ et $v \equiv 0$ satisfont $-\Delta u - u = 0 = -\Delta v - v$ dans la bande $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\}$ et $u \le v$ sur $\partial\Omega$, mais u et v ne sont pas ordonnées.

Lorsque f est décroissante sur $\mathbb{R}$, le principe de comparaison est vrai même sans l'hypothèse concernant la taille de la section de Ω. Plus précisément, nous avons:

Théorème 0.6.5 (**B.-Farina-Sciunzi**). *Supposons* $N \ge 2$ *et soit* Ω *un ouvert de* $\mathbb{R}^N$ *borné dans la direction* e_N*, le dernier vecteur de la base canonique de* $\mathbb{R}^N$*. Soit* $f : \mathbb{R} \mapsto \mathbb{R}$ *une fonction décroissante et* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *tel que:*

$$\begin{cases} -\Delta u - f(u) \le -\Delta v - f(v) & \text{dans } \mathcal{D}'(\Omega), \\ |u|, |v| \le a|x|^\delta e^{\gamma|x|} & \text{dans } \Omega, \\ u \le v & \text{sur } \partial\Omega, \end{cases}$$

où $a > 0$, $\delta \ge 0$ *et* $\gamma \in \left[0, \frac{\pi}{4\mathtt{S}_{e_N}(\Omega)\sqrt{e-1}}\right)$.
Alors,

$$u \le v \text{ dans } \Omega$$

.

L'hypothèse sur la croissance de u dans le théorème 0.6.5 est nécéssaire pour assurer la validité du principe de comparaison. En effet, considérons la fonction harmonique $u_k(x, y) = \sum_{m=1}^{k} (e^{mx} + e^{-mx}) \sin(mx)$ pour tout entier $k \ge 1$. Cette fonction s'annule sur le bord de la bande $\Omega := \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\}$ mais pourtant u_k change de signe sur Ω. Il faut donc imposer une restriction sur la croissance exponentielle des fonctions à comparer pour obtenir le résultat.

0.6.3 Applications de nos nouveaux principes de comparaison

Le rôle de nos nouveaux principes de comparaison est double:

1) Ils sont cruciaux pour prouver nos nouveaux résultats de monotonie sur des épigraphes. Ils permettent de montrer que la méthode des hyperplans mobiles peut

être initialisée indépendamment du signe de $f(0)$ et pour une large classe de domaines non bornés. Specifiquement, nous avons le résultat suivant:

Corollaire 0.6.1 (**B.-Farina-Sciunzi**). *Supposons $N \geq 2$. Soit Ω un domaine contenu dans le demi-espace $\mathbb{R}^N_+$ satisfaisant la propriété $\mathcal{R}$ que l'on rappelle ici:*

$\mathcal{R}$: Ω `contient la réflection (par rapport à l'hyperplan` $\{x_N = \theta\}$`) de toutes les bandes` $\Omega \cap \{x_N < \theta\}$.

Soit u une solution distributionnelle (*resp. classique*) (SPE). *Supposons que l'une des assertions suivante est vérifiée:*

(i) $f \in Lip_{loc}([0, +\infty))$ et u est bornée sur $\Omega \cap \{x_N < R\}$;

(ii) $f \in Lip([0, +\infty))$ (*resp. une fonction décroissante*) *et u a, au plus, une croissance exponentielle sur $\Omega \cap \{x_N < R\}$.*

Alors, il existe $R_0 \in (0, R)$ tel que $\Omega \cap \{x_N < R_0\} \not\equiv \emptyset$ et

$$\frac{\partial u}{\partial x_N} > 0 \quad dans \quad \Omega \cap \{x_N < R_0\}.$$

Ce résultat étend celui prouvé dans [Far20] pour le cas spécial du demi-espace.

Pour des raisons de clarté (et de simplicité), nous avons établis le résultat précédent dans le cas où $\Omega \subseteq \mathbb{R}^N_+$. Puisque le problème considéré est invariant par isométrie, il est clair que le même résultat reste vrai si Ω est contenu dans un demi-espace affine H. Dans ce cas, la monotonie sera alors obtenue dans la direction ν, la normale intérieure de H.

2) Ils nous permettent également de prouver des nouveaux résultats d'unicité et de symétrie pour des solutions (éventuellement non bornées et pouvant changer de signe) du problème de Dirichlet homogène de l'équation de Poisson semi-linéaire dans des domaines non bornés généraux .

Corollaire 0.6.2 (**B.-Farina-Sciunzi**). *Soit Ω et f comme dans les énoncés des théorèmes 0.6.3 (resp. 0.6.4 or 0.6.5). Alors, le problème de Dirichlet*

$$\begin{cases} -\Delta u = f(u) & dans \quad \mathcal{D}'(\Omega), \\ u = 0 & sur \quad \partial\Omega, \end{cases} \tag{0.6.10}$$

admet, au plus, une solution u satisfaisant les conditions de régularité et de croissance énoncées dans les théorèmes 0.6.3 (resp. 0.6.4 or 0.6.5).
En parliculier, si $f(0) = 0$, alors la fonction $u \equiv 0$ est l'unique solution du problème de Dirichlet (0.6.10).

Les exemples évoqués à la suite du théorème 0.6.5, montrent que le résultat précédent n'est plus vrai si les hypothèses concernant la taille de Ω ou la croissance de u ne sont pas respectées.

Une conséquence immédiate du corollaire 0.6.2 est le résultat de symétrie général pour les solutions du problème de Dirichlet homogène (0.6.10) suivant:

Corollaire 0.6.3 (B.-Farina-Sciunzi). *Soient Ω, f comme dans l'énoncé du corollaire 0.6.2 et supposons que le problème de Dirichlet*

$$\begin{cases} -\Delta u = f(u) & \textit{dans} \quad \mathcal{D}'(\Omega), \\ u = 0 & \textit{sur} \quad \partial\Omega, \end{cases} \tag{0.6.11}$$

admet une solution u satisfaisant les conditions de régularité et de croissances énoncés dans le corollaire 0.6.2.

Soit ρ une isométrie de $\mathbb{R}^N$, $N \geq 2$. Si Ω est invariant par rapport à ρ, i.e., Ω satisfait $\rho(\Omega) = \Omega$, alors la solution u hérite des mêmes symétries, c'est-à-dire, $u(x) = u(\rho(x))$ pour tout $x \in \Omega$.

Illustrons ce dernier résultat par le *problème de torsion* d'une barre solide infinie. Commençons par le cas d'une barre droite solide infinie de section sphérique. C'est-à-dire, étudions les solutions du problème:

$$\begin{cases} -\Delta u = K & \text{dans} \quad \mathcal{D}'(\Omega), \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.6.12}$$

où $\Omega = \Omega_{K,R} := \mathbb{R}^{N-K} \times B_R^K$, avec $1 \leq K < N$ un entier, et B_R^K désigne la boule de $\mathbb{R}^K$ centrée à l'origine et de rayon $R > 0$.

Puisque $\Omega_{K,R}$ est invariant par rapport à toute translation dans les variables $x_1, \ldots, x_{N-K}$ et aussi à toute rotation dans les variables $x_{N-K+1}, \ldots, x_N$, le corollaire 0.6.3 (appliqué à la fonction décroissante $f \equiv K$.) nous dit que l'unique solution u de (0.6.12), avec une croissance sous-exponentielle à l'infini, doit être indépendant de $x_1, \ldots, x_{N-K}$ et radiale dans les variables $x_{N-K+1}, \ldots, x_N$. On peut donc vérifier que la fonction

$$u(x) = \frac{R^2 - (x_{N-K+1}^2 + \ldots + x_N^2)}{2}, \qquad x \in \Omega_{K,R}$$

est la seule solution du problème (0.6.12) avec une croissance sous-exponentielle à l'infini.

L'analyse précédente s'applique aux barres solides infinies générales (pas nécessairement droites). Elle s'applique par exemple aux exemples suivants:

1) $\Omega = \mathbb{R}^{N-K} \times \omega$, où $1 \leq K < N$ est un entier et ω est un ouvert borné de $\mathbb{R}^K$. Dans ce cas, l'unique solution u de (0.6.12) avec une croissance sous-exponentielle à l'infini, dépend seulement des variables $x_{N-K+1}, \ldots, x_N$.

2) $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ -\varphi(x_1) < x_2 < \varphi(x_1)\}$, où $\varphi : \mathbb{R} \mapsto (0, +\infty)$ est une fonction continue bornée. Dans ce cas, la solution u doit être symétrique par rapport à la droite $x_2 = 0$.

3) Ω est un domaine de révolution de $\mathbb{R}^N$,$N \geq 3$, i.e., $\Omega = \left\{x \in \mathbb{R}^N \ : \ \sqrt{x_2^2 + \ldots + x_N^2} < \varphi(x_1)\right\}$, où $\varphi : \mathbb{R} \mapsto (0, +\infty)$ est une fonction continue et bornée. Ici, u doit avoir la forme $u = u(x_1, \sqrt{x_2^2 + \ldots + x_N^2})$.

Observons également que si φ est T-périodique, avec $T > 0$, alors la solution u est T-périodique dans la variable x_1.

Il est clair que les discussions et résultats ci-dessus sont également valables pour des non-linéarités f générales, satisfaisant les hypothèses du corollaire 0.6.3.

0.7 Résultats de rigidité pour le problème sur-déterminé de Serrin (SOP) dans des épigraphes

À l'aide de nos résultats de monotonie, nous pouvons apporter de nouvelles informations concernant la forme de Ω et le comportement des solutions du problème sur-déterminé de Serrin, (SOP), i.e,

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \Omega, \\ u > 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{sur} \quad \partial\Omega, \end{cases} \tag{SOP}$$

Ainsi dans cette section, des avancées concernant la conjecture D dans des épigraphes sont apportées. Dans les sous-sections suivantes, on présente d'abord les théorèmes existants sur le sujet, et ensuite on introduit nos nouveaux résultats.

0.7.1 Le problème sur-déterminé de Serrin avec une non-linéarité de type Allen-Cahn

Dans cette sous-section, on s'intéresse au problème (SOP) où f est une fonction de type Allen-Cahn (voir 0.5.1). Dans ce cadre, H. Berestycki, L.A. Caffarelli et L. Nirenberg ont prouvé le résultat suivant (voir [BCN97b]):

Théorème 0.7.1 (Berestycki-Caffarelli-Nirenberg). *Soit f une fonction de type Allen-Cahn, c'est-à-dire qui satisfait les conditions* (0.5.1). *Soit $g : \mathbb{R}^{N-1} \to \mathbb{R}$ une fonction de classe C^2 satisfaisant la condition suivante:*

$$\textit{pour tout } \tau \in \mathbb{R}^{N-1} \qquad \lim_{x \in \mathbb{R}^{N-1}, |x| \to \infty} g(x+\tau) - g(x) = 0 \tag{0.7.1}$$

et Ω son épigraphe. S'il existe une solution bornée u au problème (SOP) *alors Ω est un demi-espace supérieur et u est symétrique et strictement monotone.*

Ceci donne une réponse partielle à la conjecture D. En effet, comme le problème (SOP) est invariant par isométries euclidiennes, toutes les solutions naturelles de (SOP) correspondant aux fonctions affines $g(x) = a \cdot x + b$, $a \neq 0$, sont exclues par la condition supplémentaire sur g à l'infini.

Ce théorème est valable dans toutes les dimensions, néanmoins la preuve est basée de manière cruciale sur le fait que f est une fonction de type Allen-Cahn, ainsi que sur la condition supplémentaire (0.7.1) sur l'épigraphe à l'infini.

En 2010, A. Farina et E. Valdinoci ont introduit dans leur article [FV10] un nouvel outil géométrique qui leur permet de montrer la conjecture D sans hypothèse supplémentaire sur l'épigraphe Ω, mais sous la contrainte de dimension $N \leq 3$.

Théorème 0.7.2 (Farina-Valdinoci). *Soient $N = 2, 3$, et Ω un épigraphe de $\mathbb{R}^N$ de classe C^3 et globalement Lipschitzien. Supposons que $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ est une solution de* (SOP) *avec f une fonction de type Allen-Cahn.*
Alors, on a $\Omega = \mathbb{R}^N_+$ à une isométrie près et il existe une fonction $u_0 : (0, +\infty) \to (0, +\infty)$ tel que:

$$u(x_1, \cdots, x_N) = u_0(x_N) \text{ pour tout } (x_1, \cdots, x_N) \in \mathbb{R}^N_+.$$

Par la suite, le nouvel outil géométrique introduit dans [FV10] a été utilisé pour étudier le problème surdéterminé de Serrin sur des domaines généraux d'une variété riemannienne (voir [FMV13]).

En général, on ne peut pas espérer prouver la conjecture D en toutes dimensions car dans [PPW15], M. Del Pino, F. Pacard et J. Wei ont construit un épigraphe régulier Ω (qui n'est pas un demi-espace) tel que, si $N \geq 9$, le problème sur-déterminé (SOP) admet une solution régulière, bornée et monotone. Les auteurs notent également que cet épigraphe n'est pas globalement Lipschitzien.

Finalement, dans [WW19], K. Wang et J. Wei étudient la conjecture D, où f est une *non-linéarité de type Allen-Cahn spécifique*, c'est-à-dire lorsque f satisfait la définition suivante:

Définition 0.7.1. Une fonction f est une non linéarité de type Allen-Cahn spécifique, si la fonction

$$W(t) := \int_0^1 f(s)ds - \int_0^t f(s)ds, \qquad t \geq 0,$$

est de classe $C^2([0, +\infty))$ et satisfait les conditions suivantes:

W1) $W \geq 0$, $W(1) = 0$ et $W > 0$ dans $[0, 1)$;

W2) $W' < 0$ dans $(0, 1)$, et soit $W'(0) \neq 0$ ou

$$W'(0) = 0 \quad \text{et} \quad W''(0) \neq 0;$$

W3) $\exists\, \delta_2 \in (0, 1), \exists\, \kappa > 0$ tel que $W'' \geq \kappa$ dans $[\delta_2, 1]$;

W4) $\exists\, p > 1, \exists\, c > 0$ tel que $W'(t) \geq c(t-1)^p$ pour $t > 1$.

Le représentant typique de la classe définie ci-dessus est fourni par $f(t) = t - t^3$, la non-linéarité d'Allen-Cahn. Dans ce cas, nous avons $W(t) = \frac{(1-t^2)^2}{4}$. Nous notons également qu'une non-linéarité de type Allen-Cahn spécifique est un cas particulier d'une fonction de type Allen-Cahn (voir (0.5.1)).

Pour ce type de fonctions, K. Wang et J. Wei prouvent la conjecture D, sous différentes hypothèses sur l'épigraphe Ω; Plus précisément, ils démontrent le résultat de rigidité suivant:

Théorème 0.7.3 (**Wang-Wei**). *Supposons $N \geq 2$. Soit g une fonction globalement Lipschitzienne et Ω son épigraphe. Si le problème sur-déterminé de Serrin* (SOP) *admet une solution alors Ω est nécessairement un demi-espace et u est symétrique.*

Ce théorème est vrai en toute dimension. Cela confirme également que l'épigraphe précédemment construit par M. Del Pino, F. Pacard et J. Wei en dimension $N \geq 9$ ne peut pas être globalement Lipschitzien.

Les auteurs de [WW19] démontrent également la conjecture D en dimension $N \leq 8$, lorsque Ω est un épigraphe régulier, pas nécessairement globalement Lipschitzien, mais en imposant que la solution considérée u soit monotone.

Théorème 0.7.4 (**Wang-Wei**). *Soit Ω un épigraphe régulier et $u \in C^2(\overline{\Omega})$ une solution de* (SOP)*, tel que,*

$$\frac{\partial u}{\partial x_N} > 0, \quad \textit{dans } \Omega.$$

Si $N \leq 8$, alors Ω est un demi-espace et il existe une fonction $u_0 : [0, +\infty) \to [0, +\infty)$ tel que, $u(x) = u_0(x \cdot e)$, après une translation et e est un vecteur unitaire.

Ce résultat recouvre et complète le théorème 0.7.2, car au vu du théorème de monotonie 0.5.5 de H. Berestycki, L.A. Caffarelli and L. Nirenberg, les solutions de (SOP) dans les épigraphes globalement Lispchitziens sont strictement croissantes dans la direction x_N.

Nous sommes maintenant prêt à énoncer nos résultats. Dans notre deuxième article (voir le chapitre 2), nous établissons également de nouveaux résultats de rigidité pour des non-linéarités de type Allen-Cahn spécifiques. Plus précisément, nous prouvons:

Théorème 0.7.5 (**B.-Farina**). *Supposons $N \geq 2$ et soit $f \in C^1([0, +\infty))$ une fonction de type Allen-Cahn spécifique. Soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de* (SOP) *satisfaisant*

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega. \tag{0.7.2}$$

Soit $\Omega \subset \mathbb{R}^N$ un épigraphe définit par une fonction $g \in C^1(\mathbb{R}^{N-1})$ et, si $N \geq 9$, supposons également que g satisfait:

a) $\exists\, C > 0$ *pour lequel:*

$$g(x') \geq -C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1};$$

ou

b) $\exists\, C > 0$ *pour lequel:*

$$g(x') \leq C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1}.$$

Alors, $\Omega = \mathbb{R}^N_+$ à une isométrie près et il existe $u_0 : [0, +\infty) \to [0, +\infty)$ strictement croissante tel que:

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+.$$

Les remarques suivantes s'imposent:

(i) Dans le théorème ci-dessus, nous ne supposons pas que u est borné. En effet, cette propriété est automatiquement satisfaite par toute solution de (SPE) sur des domaines généraux, éventuellement non bornés, lorsque f est une non-linéarité de type Allen-Cahn spécifique. Voir le corollaire 2.5.1.

(ii) Lorsque $N \leq 8$, le théorème ci-dessus recouvre le théorème 0.7.4 présenté au-dessus. On remarque également que, lorsque g est globalement Lipschizienne, l'hypothèse de monotonie (0.7.2) est automatiquement satisfaite en toute dimension $N \geq 2$ grâce au point (i) et au théorème 0.5.5. Comme le caractère Lipschitzien de g implique la validité de la condition a) et b), nous voyons que le théorème ci-dessus recouvre également le théorème 0.7.3. Lorsque g est coercive, l'hypothèse de monotonie (0.7.2) est automatiquement satisfaite en toute dimension $N \geq 2$ grâce au théorème 0.4.4 de M.J Esteban et P.L. Lions et, comme la condition supplémentaire a) est clairement satisfaite, on voit aussi que le théorème 1.3 de [WW19] est un cas particulier du théorème 0.7.4 ci-dessus.

(iii) Les hypothèses supplémentaires pour $N \geq 9$ ont l'interprétation géométrique suivante: soit $\mathbb{R}^N \setminus \Omega$ contient un cône, soit Ω contient un cône (ou de manière équivalente, soit le graphe de g se trouve au-dessus du graphe d'un cône, soit le graphe de g se trouve en dessous du graphe d'un cône).

(iv) Les remarques ci-dessus montrent aussi que l'épigraphe construit par M. Del Pino, F. Pacard et J. Wei dans [PPW15] ne contient pas de cône (et son complémentaire non plus) et donc que cet épigraphe n'est ni globalement Lipschitzien (comme déjà observé dans [PPW15]), ni contenu dans un demi-espace affine.

Si nous supposons en outre que Ω est un épigraphe suffisamment régulier et minoré, alors la condition de monotonie (0.7.2) est automatiquement satisfaite, grâce au théorème 0.5.17 démontré dans notre premier article. Ainsi, nous avons prouvé le résultat suivant:

Théorème 0.7.6 (B.-Farina). *Supposons $N \geq 2$ et soit Ω un épigraphe minoré définit par une fonction $g \in \mathcal{G} \cap C^1(\mathbb{R}^{N-1})$. Soit $f \in C^1([0, +\infty))$ une non-linéarité de type Allen-Cahn spécifique et soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de* (SOP).

Alors, $\Omega = \mathbb{R}^N_+$; à une translation verticale près et il existe $u_0 : [0, +\infty) \to [0, +\infty)$ strictement croissante, tel que:

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+. \tag{0.7.3}$$

Par conséquent, le théorème (0.7.6) prouve la conjecture D pour une large famille d'épigraphes continus et minorés, pas nécessairement globalement Lipschitzien, lorsque f est une fonction de type Allen-Cahn spécifique. Une question naturelle est de savoir si le même résultat peut être obtenu pour une fonction f plus générale.

0.7.2 Le problème sur-déterminé de Serrin avec une non-linéarité générale

Dans la littérature mathématique, il existe peu de résultats concernant le problème (SOP) et la conjecture D dans le cas où f est une fonction localement Lipschitzienne générale.

Dans ce contexte, A. Ros, D. Ruiz et P. Sicbaldi (voir [RRS17]) résolvent la conjecture D en dimension $N = 2$ pour des épigraphes suffisamment réguliers. Précisément, ils montrent le résultat suivant:

Théorème 0.7.7 (Ros-Ruiz-Sicbaldi). *Soit $g : \mathbb{R} \to \mathbb{R}$ une fonction de classe $C^{1,\alpha}$ et Ω son épigraphe. Supposons qu'il existe une solution u bornée au problème* (SOP) *où $f : [0, +\infty) \to \mathbb{R}$ est une fonction localement Lipschitzienne. Alors Ω est un demi-plan et u ne dépend que d'une variable.*

Notons que le théorème énoncé dans [RRS17] est plus général car il s'applique à des domaines réguliers Ω tel que $\partial\Omega$ est non borné et connexe.

Dans tous les résultats présentés ci-dessus, ainsi que dans la conjecture D, les solutions du problème (SOP) sont toujours supposées bornées (et ce fait est important pour effectuer les preuves). Un premier résultat de rigidité concernant les solutions non bornées du problème (SOP) est établi dans [FV10]. La nécessité de considérer également des solutions non bornées au problème sur-déterminé de Serrin (SOP) vient du fait que, pour certaines fonctions naturelles f, les seules solutions à (SOP) sont non bornées. Ceci est bien illustré par le cas des fonctions harmoniques, c'est-à-dire lorsque $f \equiv 0$. Dans ce cas, il pourrait ne pas y avoir de solution bornée à (SOP), alors que la fonction $u(x) = x_N$ est une fonction harmonique positive non bornée sur le demi-espace $\mathbb{R}^N_+$, avec une condition de Dirichlet nulle.

Dans ce contexte, les auteurs de [FV10] ont établi un résultat de rigidité pour des solutions éventuellement non bornées de (SOP) en dimension 2:

Théorème 0.7.8 (Farina-Valdinoci). *Soit $g : \mathbb{R} \to \mathbb{R}$ une fonction de classe C^3 et Ω son épigraphe. Soit f une fonction localement Lipschitzienne et $u \in C^2(\overline{\Omega})$ une solution de* (SOP)*, tel que $|\nabla u| \in L^\infty(\Omega)$. Supposons que u satisfait la condition suivante:*

$$\frac{\partial u}{\partial x_2} > 0, \quad \textit{dans } \Omega.$$

Alors, Ω doit être une demi-plan, et il existe $\omega \in \mathbb{S}^1$ et une fonction $u_0 : \mathbb{R} \to (0, +\infty)$ tel que ω est normale à $\partial\Omega$ et $u(x) = u_0(\omega \cdot x)$ pour tout $x \in \Omega$.

Pour prouver leur théorème, A. Farina et E. Valdinoci ont développé une nouvelle approche géométrique du problème (SOP) qui permet d'obtenir un résultat de rigidité pour des domaines Ω connexes et de classe C^3 (pas nécessairement des épigraphes).

Nous sommes maintenant prêts à présenter nos contributions concernant le problème (SOP) pour une fonction générale f. Elles ont été obtenues dans notre deuxième article (voir le chapitre 2), où nous avons démontré de nouveaux résultats de rigidité pour des solutions potentiellement non bornées de (SOP) dans des épigraphes minorés et en dimensions $N \leq 3$.

Les preuves des résultats suivants sont obtenues en combinant l'approche géométrique développée dans [FV10] et les résultats de monotonie décrits dans la première section (voir 0.1).

Nous allons commencer par un résultat de rigidité pour les solutions bornées du problème sur-déterminé (SOP) dans un épigraphe globalement Lipschitzien et minoré de $\mathbb{R}^3$:

Théorème 0.7.9 (B.-Farina). *Soit $\Omega \subset \mathbb{R}^3$ un épigraphe globalement Lipschitzien minoré avec un bord de classe C^3. Soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution bornée de* (SOP)*, où $f \in C^1([0, +\infty))$ satisfait $f(0) \geq 0$ et*

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t), \tag{0.7.4}$$

ici F désigne la primitve de f s'annulant en 0.
Alors, $\Omega = \mathbb{R}^3_+$ à une translation verticale près et il existe $u_0 : [0, +\infty) \to (0, +\infty)$ strictement croissante tel que:

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Notons que l'hypothèse (0.7.4) est naturelle et, d'une certaine manière, nécessaire. En effet, si la conjecture D est vraie, c'est-à-dire si Ω est un demi-espace affine, alors le principe de continuation unique (voir le théorème 1 dans [FV13]) garantit que la solution u doit être unidimensionnelle. Ensuite, une analyse standard de l'intégrale première donne que $F(\sup u) > F(t)$, pour tout $t \in [0, \sup u)$, confirmant ainsi la validité de (0.7.4).

De plus, la condition (0.7.4) est clairement satisfaite dès que:

(a) $f(t) \geq 0$, pour tout $t \geq 0$,

(b) il existe $\zeta > 0$, tel que $f(t) \geq 0$ pour tout $t \in [0, \zeta]$ et $f(t) \leq 0$ pour tout $t \in [\zeta, +\infty)$.

Notons que le théorème 0.7.9 est également vrai en dimension 2 sans condition (0.7.4), même pour les solutions non bornées de (SOP). Ceci est contenu dans le théorème suivant (voir point H1).

Théorème 0.7.10 (B.-Farina). *Soit $\Omega \subset \mathbb{R}^2$ un épigraphe globalement Lipschitzien et minoré avec un bord de classe C^3 et soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de* (SOP)*.*

Supposons que l'une des assertions suivantes soit vraie :

$$f \in Lip_{loc}([0,+\infty)), f(0) \geq 0 \;\; et \;\; \nabla u \in L^{\infty}(\Omega); \tag{H1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad f(0) \geq 0, \quad \exists\, \zeta > 0 \,:\, f(t) \leq 0 \quad dans \quad [\zeta,+\infty), \\ \exists\, \alpha \in [0,1) \,: \quad u(x) = O(|x|^{\alpha}) \qquad quand \quad |x| \longrightarrow \infty; \end{cases} \tag{H2}$$

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(0) \geq 0, \quad \exists\, \zeta > 0 \,:\, f(t) \geq 0 \quad dans \quad [\zeta,+\infty), \\ u(x) = o(\ln|x|), \qquad quand \quad |x| \longrightarrow \infty; \end{cases} \tag{H3}$$

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(0) \geq 0, \quad f(t) \leq 0 \quad dans \quad (0,+\infty), \\ u(x) = o(|x| \ln^{\frac{1}{2}}|x|), \qquad quand \quad |x| \longrightarrow \infty. \end{cases} \tag{H4}$$

Alors, $\Omega = \mathbb{R}^2_+$ *à une translation verticale près et il existe* $u_0 : [0,+\infty) \to (0,+\infty)$ *strictement croissante tel que:*

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

Nous avons également établi un résultat de rigidité pour des solutions éventuellement non bornées de (SOP) en dimension 3.

Théorème 0.7.11 (B.-Farina). *Soit* $\Omega \subset \mathbb{R}^3$ *un épigraphe globalement Lipschitzien et minoré de classe* C^3 *et soit* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *une solution de* (SOP)*.*

Supposons que l'une des assertions suivantes soit vérifiée:

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(t) \geq 0 \quad pour\ tout \quad t \geq 0, \\ u(x) = o(\ln|x|), \qquad quand \quad |x| \longrightarrow \infty; \end{cases} \tag{A1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad \exists\, \zeta \geq 0 \,:\, f(t) \geq 0 \quad sur \quad [0,\zeta] \quad et \quad f(t) \leq 0 \quad sur \quad [\zeta,+\infty), \\ \exists\, \alpha \in [0,1) \,: \quad u(x) = O(|x|^{\alpha}) \qquad quand \quad |x| \longrightarrow \infty; \end{cases} \tag{A2}$$

Alors, $\Omega = \mathbb{R}^3_+$ *à une translation verticale près et il existe* $u_0 : [0,+\infty) \to (0,+\infty)$ *strictement croissante tel que:*

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Il s'agit du premier résultat de rigidité établi pour des solutions éventuellement non bornées de (SOP) dans des épigraphes inclus dans $\mathbb{R}^3$.

Les preuves des théorèmes précédents sont essentiellement basées sur le théorème de rigidité suivant, qui provient lui-même de l'utilisation d'une formule de type Poincaré et d'une analyse détaillée des ensembles de niveaux des solutions (voir [FV10])):

Théorème 0.7.12 (B.-Farina). *Supposons $N = 2$ ou 3 et soit $\Omega \subset \mathbb{R}^N$ un domaine de classe C^3. Soient $f \in Lip_{loc}([0, +\infty))$ et $u \in C^2(\overline{\Omega})$ une solution de* (SOP) *tel que:*

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad \textit{quand } R \longrightarrow \infty. \tag{0.7.5}$$

Si u est monotone, i.e.,

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{0.7.6}$$

alors, $\Omega = \mathbb{R}^N_+$ à isométrie près et il existe $u_0 : [0, +\infty) \to (0, +\infty)$ strictement croissante tel que:

$$u(x) = u_0(x_N) \quad \forall x \in \mathbb{R}^N_+.$$

Notons que ce théorème reste vrai en toutes dimensions, cependant la condition d'énergie (0.7.5) est moins naturelle (et plus difficile à satisfaire) lorsque $N \geq 3$.

Grâce à notre nouveau résultat de monotonie 0.5.15 établi dans un épigraphe suffisamment régulier et minoré avec $\mathfrak{c} \neq 0$, les théorèmes 0.7.9 0.7.10 et 0.7.11 sont toujours vrais si l'on suppose que $f(0) < 0$.

Théorème 0.7.13 (B.-Farina). *Soit $\Omega \subset \mathbb{R}^2$ un épigraphe globalement Lipschitzien minoré et définit par une fonction $g \in C^3(\mathbb{R}) \cap \mathbb{C}^{1,\gamma}(\mathbb{R})$.*
Soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de (SOP) *avec $\mathfrak{c} \neq 0$ et supposons que l'une de assertions suivante est vérifiée:*

$$f \in Lip_{loc}([0, +\infty)), f(0) < 0 \textit{ et } \nabla u \in L^\infty(\Omega); \tag{H1}$$

$$\begin{cases} f \in Lip_{loc}([0, +\infty)), \quad f(0) < 0, \quad \exists\, \zeta > 0 \,:\, f(t) \leq 0 \quad \textit{dans} \quad [\zeta, +\infty), \\ \exists\, \alpha \in [0, 1) \,: \quad u(x) = O(|x|^\alpha) \qquad \textit{comme} \quad |x| \longrightarrow \infty; \end{cases} \tag{H2}$$

Alors, $\Omega = \mathbb{R}^2_+$ à une translation verticale près et il existe $u_0 : [0, +\infty) \to (0, +\infty)$ strictement croissante tel que:

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

Théorème 0.7.14 (B.-Farina). *Soit $\Omega \subset \mathbb{R}^3$ un épigraphe globalement Lipschitzien minoré et définit par une fonction $g \in C^3(\mathbb{R}^2) \cap \mathbb{C}^{1,\gamma}(\mathbb{R}^2)$.*
Soit $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ une solution de (SOP) *avec $\mathfrak{c} \neq 0$.*
Soit $f \in C^1([0, +\infty))$ tel que $f(0) < 0$ et

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t).$$

Alors, $\Omega = \mathbb{R}^3_+$ à une translation verticale près et il existe $u_0 : [0, +\infty) \to (0, +\infty)$ strictement croissante tel que:

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

0.8 Résultats de classification pour les solutions de l'équation de Poisson semi-linéaire dans le demi-espace

Dans cette section, nous présentons une classification des solutions éventuellement non bornées du problème suivant:

$$\begin{cases} -\Delta u = f(u) & \text{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{dans} \quad \mathbb{R}^N_+, \\ u = 0 & \text{sur} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{SPE+}$$

où f est une fonction localement Lipschitzienne sur $[0, +\infty)$.

Comme dans les sections précédentes, nous présentons d'abord les travaux existants concernant ce problème, puis nous introduisons nos nouveaux résultats qui sont contenus dans notre troisième article (voir le chapitre 3).

Étudier et classifier les solutions du problème (SPE+) est naturel pour différentes raisons:

(i) Le demi-espace est le domaine non borné le plus simple avec un bord non borné dans lequel on peut étudier le problème de Dirichlet pour l'équation de Poisson semi-linéaire.

(ii) Le problème (SPE+) peut apparaître comme un problème limite, après une procédure d'explosion ("blow-up" en Anglais) près de la frontière ou lorsque l'on veut mettre en évidence des estimées *à priori* pour des solutions strictement positives d'un problème semi-linéaire uniformément élliptique dans un domaine borné régulier ([GS81a]), mais également, lorsque l'on étudie l'unicité des solutions classiques d'un problème perturbé comme suit:

$$-\varepsilon^2 \Delta v = f(v) \quad \text{dans} \quad \Omega, \qquad v > 0 \quad \text{dans} \quad \Omega, \quad v = 0 \quad \text{sur} \quad \partial\Omega, \tag{0.8.1}$$

où Ω est un domaine régulier et borné, f est une fonction régulière et ε est un petit paramètre positive (voir par exemple [Ang85] et [Dan09]).

Notons que, lorsque $f(0) < 0$, même si les solutions v considérées dans (0.8.1) sont strictement positives, étant donné l'absence du principe du maximum fort, la solution u de (SPE+) obtenue après la procédure de mise à l'échelle est "seulement" positive dans $\mathbb{R}^N_+$.

Ceci motive le choix de considérer les solutions positives de (SPE+) (et pas seulement strictement positives).

Enfin, la nécessité de considérer également les solutions non bornées de (SPE+) est bien illustrée par le cas des fonctions harmoniques, c'est-à-dire lorsque $f \equiv 0$. Dans ce cas, il est bien connu que toutes les solutions de (SPE+) sont données par $u(x) = \alpha x_N$, $\alpha \geq 0$.

Le premier résultat concernant la classification complète des solutions de (SPE+) avec $f \not\equiv 0$ a été obtenu par B. Gidas et J. Spruck en 1981. Il concerne la fonction $f(u) = u^p$

et il a été motivé par la situation décrite au point (ii) ci-dessus.

Théorème 0.8.1 (**Gidas-Spruck**). *Supposons* $N \geq 2$ *et soit* p *un réel satisfaisant* $1 < p \leq \frac{N+2}{N-2}$. *La seule solution classique de*

$$\begin{cases} -\Delta u = u^p & dans \quad \mathbb{R}^N_+, \\ u \geq 0 & dans \quad \mathbb{R}^N_+, \\ u = 0 & sur \quad \partial\mathbb{R}^N_+, \end{cases}$$

est $u \equiv 0$.

La preuve dans [GS81a] utilise la méthode des sphères mobiles ("moving spheres method" en Anglais). En 1993, H. Berestycki, L.A. Caffarelli et L. Nirenberg ([BCN93]) ont observé que la preuve dans [GS81a] donne également le résultat suivant:

Théorème 0.8.2 (**Berestycki-Caffarelli-Nirenberg**). *Soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution de* (SPE+) *où* $f \in Lip_{loc}([0, +\infty))$. *Si l'une des conditions suivante est vérifiée:*

(i) $N = 2$ *et* $f \geq 0$,

(ii) $N \geq 3$ *et* f *satisfait*

$$t \mapsto \frac{f(t)}{t^{(N+2)/(N-2)}} \quad \textit{est décroissante sur } (0, +\infty). \tag{0.8.2}$$

Alors, soit $u \equiv 0$, *ou il existe une fonction* $u_0 : [0, +\infty) \to [0, +\infty)$ *tel que:*

$$u(x) = u_0(x_N) \qquad \textit{pour tout } x \in \mathbb{R}^N_+ \tag{0.8.3}$$

et $u_0 > 0$, $u_0' > 0$ *sur* $(0, +\infty)$.

Notons que l'on a nécessairement $f(0) \geq 0$ dans le théorème ci-dessus.

Les solutions de la forme (0.8.3), c'est-à-dire, les solutions dépendant seulement de la variable x_N, sont dites *unidimensionnelles* puisqu'elles héritent des symétries de $\mathbb{R}^N_+$. Clairement, le profil u_0 est solution du problème

$$\begin{cases} -u_0'' = f(u_0) & \text{dans} \quad (0, +\infty), \\ u_0(0) = 0. \end{cases} \tag{0.8.4}$$

Récemment, les résultats suivants ont été obtenus concernant la symétrie unidimensionnelle des solutions de (SPE+). Ils couvrent également le cas où $f(0) < 0$.

Théorème 0.8.3 (**Farina-Sciunzi**). *Soit* $u \in C^2(\overline{\mathbb{R}^2_+})$ *une solution de* (SPE+) *où* $f \in Lip_{loc}([0, +\infty))$. *Alors,*

(i) si $f(0) \geq 0$, *alors soit* $u \equiv 0$, *ou* u *est strictement positive sur* $\mathbb{R}^2_+$ *et* $\frac{\partial u}{\partial x_2} > 0$ *sur* $\mathbb{R}^2_+$;

(ii) *si* $f(0) < 0$*, alors soit* $u > 0$ *et* $\frac{\partial u}{\partial x_2} > 0$ *sur* $\mathbb{R}^2_+$*, ou*

$$u(x) = u_0(x_2) \qquad \textit{pour tout } x \in \mathbb{R}^2_+$$

où $u_0 : [0, +\infty) \to [0, +\infty)$ *est une fonction unique et périodique;*

(iii) *si pour un* $p > 1$*,*

$$\lim_{t \to +\infty} \frac{f(t)}{t^p} \in (0, +\infty), \tag{0.8.5}$$

alors u *est bornée et unidimensionnelle.*

(iv) *si* $f \in C^1([0, +\infty))$ *avec* $f(0) < 0$ *et* $f'(t) \geq \alpha > 0$ *pour tout* $t > 0$*, alors* u *est unidimensionnelle et périodique.*

Théorème 0.8.4 (Cortázar-Elgueta-García-Melián). *Supposons* $N \geq 2$*. La seule solution classique de*

$$\begin{cases} -\Delta u = u - 1 & \textit{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ u = 0 & \textit{sur} \quad \partial\mathbb{R}^N_+, \end{cases}$$

est la fonction

$$u(x) = 1 - \cos(x_N) \quad \textit{pour tout } x \in \mathbb{R}^N_+.$$

Le cas $N = 2$ dans le théorème 0.8.4 est dû à [FS16] (Il suffit d'applique le point (iv) du théorème 0.8.3 avec $f(u) = u - 1$), tandis que le cas $N \geq 3$ est prouvé dans [CEG16].

Il est intéressant de noter que les résultats de symétrie unidimensionnelle dans les quatre théorèmes ci-dessus ont été obtenus sans aucune hypothèse supplémentaire sur les solutions u. Il semble donc naturel de chercher à élargir la classe des fonctions f garantissant que les seules solutions classiques, bornées ou non, de (SPE+) soient nécessairement unidimensionnelles. D'autre part, un tel résultat de symétrie unidimensionnelle n'est pas vrai pour toute fonction localement Lipschitzienne f, comme le montre la fonction $u(x) = x_N e^{-x_1}$, qui est solution de (SPE+) avec $f(u) = -u$ et $N \geq 2$. Ainsi, les deux axes de recherche suivants émergent naturellement:

(1) fournir des conditions naturelles générales sur f garantissant que les seules solutions classiques, bornées ou non, à (SPE+) sont nécessairement unidimensionnelles ;

(2) fournir des hypothèses suffisamment générales sur les solutions classiques de (SPE+) assurant leur symétrie unidimensionnelle (soit indépendamment de la fonction considérée $f \in Lip_{\mathrm{loc}}([0, +\infty))$, soit pour une sous-classe appropriée de fonctions localement Lipschitzienne).

Les nouveaux résultats présentés dans cette section sont consacrés à ces deux axes de recherche. Avant d'énoncer nos résultats, passons en revue ceux qui existent dans la littérature mathématique.

Concernant le deuxième axe de recherche, diverses hypothèses supplémentaires sur u ont été considérées dans la littérature afin d'établir la symétrie unidimensionnelle de u. Elles concernent principalement les conditions de croissance de u ou ses propriétés qualitatives ou spectrales, telles que la monotonie, la stabilité et/ou l'indice de Morse. Ces conditions sont naturelles, car elles sont généralement satisfaites par les solutions qui apparaissent dans les applications et par celles construites à l'aide de méthodes variationnelles ou topologiques.

À cet égard, nous présentons ci-dessous les principales contributions de la littérature. Nous les énoncerons par ordre croissant de généralité. Commençons par le cas des solutions bornées (voir [BCN97a]).

Théorème 0.8.5 (**Berestycki-Caffarelli-Nirenberg**). *Supposons* $N = 2$ *ou* 3*. Soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution strictement positive et bornée de* (SPE+) *où* $f \in Lip_{loc}([0, +\infty))$*. Si* $N = 2$*, alors* u *est unidimensionnelle et strictement croissante. Si* $N = 3$ *la conclusion reste vrai, si on suppose, en plus, que* $f(0) \geq 0$ *et que* f *est de classe* C^1*.*

Plus récemment, en combinant le théorème 0.8.3 ci-dessus et un outil géométrique introduit dans [FV10], les auteurs de [FS16] ont amélioré le résultat du théorème 0.8.5 dans le cas bidimensionnel. En effet, les auteurs de [FS16] démontrent la symétrie unidimensionnelle pour toute solution strictement positive, en supposant seulement que u possède un gradient borné. Plus précisément:

Théorème 0.8.6 (**Farina-Sciunzi**). *Soit* $u \in C^2(\overline{\mathbb{R}^2_+})$ *une solution de* (SPE+) *où* $f \in Lip_{loc}([0, +\infty))$*. Si* u *a un gradient borné, alors elle est unidimensionnelle.*

Rappelons que les solutions bornées de (SPE+) ont un gradient borné, mais l'inverse n'est pas vrai, comme le montre la fonction harmonique (unidimensionnelle) $u(x) = x_N$.

Pour conclure sur le cas des solutions bornées de (SPE+), nous mentionnons les deux résultats suivants. Le premier concerne les fonctions convexes f avec $f(0) = 0$, tandis que le second s'applique, entre autres, à toute fonction f strictement positive (voir [CLZ14] et [DF22]).

Théorème 0.8.7 (**Chen-Lin-Zou**). *Supposons* $N \geq 2$ *et soit* $f \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ *avec* $f(0) = 0$ *et* $f'' \geq 0$*. la seule solution classique bornée de* (SPE+) *est* $u \equiv 0$*.*

La conséquence immédiate de ce résultat est le corollaire suivant:

Corollaire 0.8.1 (**Chen-Lin-Zou**). *Supposons* $N \geq 2$ *et soit* $p > 1$*. La seule solution classique bornée de*

$$\begin{cases} -\Delta u = u^p & \text{dans } \mathbb{R}^N_+, \\ u \geq 0 & \text{dans } \mathbb{R}^N_+, \\ u = 0 & \text{sur } \partial\mathbb{R}^N_+, \end{cases}$$

est $u \equiv 0$*.*

Le corollaire 0.8.1 a d'abord été prouvé pour $1 < p < (N + 1)/(N - 3)^+$ dans [Dan92], et ensuite pour tout $1 < p < p_{JL}(N)$ dans [Far07], où p_{JL} est l'exposant de stabilité de Jospeh-Lundgren donné par $p_{JL}(N) = \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)^+}$.

Théorème 0.8.8 (**Dupaigne-Farina**). *Soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution bornée de* (SPE+). *Supposons* $f \in C^1([0, +\infty))$ *et*

1. *soit* $f(t) \geq 0$ *pour* $t \geq 0$,
2. *ou il existe* $z > 0$ *tel que* $f(t) \geq 0$ *pour* $t \in [0, z]$ *et* $f(t) \leq 0$ *pour* $t \geq z$.

Si $2 \leq N \leq 11$, *alors soit* $u \equiv 0$, *ou* u *est strictement positive, unidimensionnelle et strictement croissante.*

Pour continuer, rappelons qu'une solution de (SPE+) qui est soit strictement monotone dans la direction x_N, soit un minimiseur local est automatiquement *stable*, ce qui signifie qu'elle satisfait:

$$\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 - f'(u)\varphi^2 \geq 0, \tag{0.8.6}$$

pour tout $\varphi \in C^1_c(\mathbb{R}^N_+)$ (ici on a supposé $f \in C^1$).
Il faut également préciser que u est *stable en dehors d'un compact* $\mathcal{K} \subset \mathbb{R}^N_+$ si l'inégalité (0.8.6) est vraie pour tout $\varphi \in C^1_c(\mathbb{R}^N_+ \setminus \mathcal{K})$. On voit, clairement, que la stabilité en dehors d'un compact implique la stabilité (en fait, toute solution d'indice de Morse fini à (SPE+) est stable en dehors d'un ensemble compact de $\mathbb{R}^N_+$).

Pour ces classes de solutions de (SPE+), nous avons (voir [DSS22]):

Théorème 0.8.9 (**Dupaigne-Sirakov-Souplet**). *Supposons* $N \geq 2$. *Soit* $f \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ *une fonction positive et convexe, avec* $f(0) = 0$ *et* $f \not\equiv 0$.
Soit u *une solution classique de* (SPE+) *qui est monotone dans la direction* x_N, *i.e., tel que* $\frac{\partial u}{\partial x_N} \geq 0$ *sur* $\mathbb{R}^N_+$. *Alors,* $u \equiv 0$.
En particulier, $u \equiv 0$ *est la seule solution classique de* (SPE+) *qui est bornée sur les bandes finies.*[21]

Une conséquence immédiate du théorème 0.8.9 est la suivante:

Corollaire 0.8.2 (**Dupaigne-Souplet-Sirakov**). *Supposons* $N \geq 2$ *et soit* $p > 1$. *Soit* u *une solution classique de*

$$\begin{cases} -\Delta u = u^p & \textit{dans} \;\; \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \;\; \mathbb{R}^N_+, \\ u = 0 & \textit{sur} \;\; \partial\mathbb{R}^N_+, \end{cases} \tag{0.8.7}$$

qui est monotone dans la direction x_N. *Alors,* $u \equiv 0$.
En particulier, $u \equiv 0$ *est la seule solution classique de* (0.8.7) *qui est bornée sur les bandes finies.*

Le corollaire 0.8.2 recouvre et complète le corollaire 0.8.1.

Trés récemment, le corollaire 0.8.2 a été complété dans le papier [DFP23]. Plus précisément:

[21] Lorsque $f(0) \geq 0$, toute solution strictement positive de (SPE+), qui est bornée sure les bandes finies, est strictement monotone dans la direction x_N (voir le théorème 0.5.3 dans [Far20]).

Théorème 0.8.10 (Dupaigne-Farina-Petitt). *Supposons $N \geq 2$ et soit $p > 1$. La seule solution classique de*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{dans } \mathbb{R}^N_+, \\ u = 0 & \text{sur } \partial\mathbb{R}^N_+, \end{cases}$$

qui est stable en dehors d'un compact de $\mathbb{R}^N_+$ est $u \equiv 0$.

Notons que le théorème 0.8.10 est vrai pour des solutions pouvant changer de signe. Il a d'abord été démontré dans [Far07] pour tout $N \leq 10$ et, si $N > 10$, pour $1 < p < p_{JL}(N) = \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}$, l'exposant de stabilité de Joseph-Lundgren.

Nous sommes maintenant prêts à présenter nos résultats. Le premier théorème contribue à la première ligne de recherche ci-dessus. Il reprend et complète le résultat énoncé au point (iii) du théorème 0.8.3.

Théorème 0.8.11 (B.-Farina). *Soit $u \in C^2(\overline{\mathbb{R}^2_+})$ une solution de (SPE+) où $f \in Lip_{loc}([0,+\infty))$ satisfait*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0. \tag{0.8.8}$$

Alors, il existe une fonction bornée $u_0 : [0,+\infty) \to [0,+\infty)$ tel que:

$$u(x) = u_0(x_2) \quad \textit{pour tout } x \in \mathbb{R}^2_+.$$

De plus, soit u_0 est périodique (éventuellement égale à zéros) ou, $u_0 > 0$ et $u_0' > 0$ sur $(0,+\infty)$.

Nous présentons ensuite nos résultats concernant le deuxième axe de recherche ci-dessus. Commençons par le cas des solutions monotones dans la direction x_N.

Théorème 0.8.12 (B.-Farina). *Supposons $2 \leq N \leq 9$ et soit $u \in C^2(\overline{\mathbb{R}^N_+})$ une solution de (SPE+) tel que $\frac{\partial u}{\partial x_N} \geq 0$ sur $\mathbb{R}^N_+$. Supposons que $f \in C^1([0,+\infty))$ est une fonction positive satisfaisant:*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0. \tag{0.8.9}$$

Alors, soit $u \equiv 0$, soit il existe une fonction bornée $u_0 : [0,+\infty) \to [0,+\infty)$ tel que:

$$u(x) = u_0(x_N) \qquad \textit{pour tout } x \in \mathbb{R}^N_+$$

et $u_0 > 0$, $u_0' > 0$ sur $(0,+\infty)$.
De plus, si $f(t) > 0$ pour tout $t > 0$, alors, nécessairement $u \equiv 0$ dans $\mathbb{R}^N_+$ et $f(0) = 0$.

Bien que soumis à une restriction dimensionnelle, le résultat précédent étend le théorème 0.8.7 à toutes les fonctions f positives sous la seule hypothèse que f se comporte au moins linéairement à l'infini. Il serait intéressant de savoir si ce résultat est toujours valable pour $N > 9$.

Si l'on suppose en outre que f est convexe ou que f présente au plus une croissance polynomiale alors les conclusions du théorème 0.8.12 restent valables jusqu'à la dimension $N = 10$. Plus précisément:

Théorème 0.8.13 (B.-Farina). *Supposons* $2 \le N \le 10$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution de* (SPE+) *tel que* $\frac{\partial u}{\partial x_N} \ge 0$ *sur* $\mathbb{R}^N_+$. *Supposons que* $f \in C^1([0, +\infty))$ *est une fonction positive et convexe satisfaisant:*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{0.8.10}$$

Alors, soit $u \equiv 0$, *soit il existe une fonction bornée* $u_0 : [0, +\infty) \to [0, +\infty)$ *tel que:*

$$u(x) = u_0(x_N) \qquad \textit{pour tout } x \in \mathbb{R}^N_+$$

et $u_0 > 0$, $u_0' > 0$ *sur* $(0, +\infty)$.
De plus, si $f(0) = 0$, *alors nécessairement* $u \equiv 0$ *dans* $\mathbb{R}^N_+$.

Contrairement au théorème 0.8.7, la fonction f dans le théorème 0.8.13 n'est pas nécessairement croissante. En effet, ce dernier s'applique à $f(t) = |t-1|^p$, $p > 1$.

On note également que si on retire l'hypothèse (0.8.10), alors la conclusion du théorème 0.8.13 n'est plus vraie. En effet, la fonction *non bornée* et monotone $u(x) = \ln(1 + x_N)$ est solution du problème (SPE+) avec $f(t) = e^{-2t}$, qui est strictement positive et convexe.

Cette observation s'applique également au théorème 0.8.12 ci-dessus et au théorème suivant.

Théorème 0.8.14 (B.-Farina). *Supposons* $2 \le N \le 10$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution de* (SPE+) *tel que:* $\frac{\partial u}{\partial x_N} \ge 0$ *sur* $\mathbb{R}^N_+$. *Supposons que* $f \in C^1([0, +\infty))$ *est une fonction positive satisfaisant*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0, \qquad \limsup_{t \to +\infty} \frac{f(t)}{t^\theta} < +\infty, \tag{0.8.11}$$

pour un $\theta > 1$.
Alors, soit $u \equiv 0$, *soit il existe une fonction bornée* $u_0 : [0, +\infty) \to [0, +\infty)$ *tel que*

$$u(x) = u_0(x_N) \qquad \textit{pour tout } x \in \mathbb{R}^N_+$$

et $u_0 > 0$, $u_0' > 0$ *sur* $(0, +\infty)$.
De plus, si $f(t) > 0$ *pour tout* $t > 0$, *alors nécessairement* $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f(0) = 0$.

En conséquence immédiate des trois théorèmes précédents, nous pouvons traiter le cas de solutions bornées sur des bandes finies.

Corollaire 0.8.3 (B.-Farina). *Soient* N, f *comme dans les énoncés du théorème 0.8.12 (resp. du théorème 0.8.13 ou du théorème 0.8.14) et soit* u *une solution classique de* (SPE+) *qui est bornée sur les bandes.*
Alors les conclusions du théorème 0.8.12 (resp. du théorème 0.8.13 ou du théorème 0.8.14) restent vraies.

Le point de vue adopté pour démontrer les théorèmes précédents s'applique aussi bien aux solutions stables qu'aux solutions stables en dehors d'un compact. Nous présentons ci-dessous les théorèmes de type Liouville correspondant à ces deux classes de solutions.

Théorème 0.8.15 (**B.-Farina**). *Supposons* $2 \le N \le 4$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution stable de* (SPE+) *où* $f \in C^1([0,+\infty))$ *est une fonction positive et croissante satisfaisant*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{0.8.12}$$

Alors, $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f(0) = f'(0) = 0$.

Dans le résultat suivant, nous remplaçons la condition de superlinéarité (0.8.12) sur f par une hypothèse de croissance sur la solution u.

Théorème 0.8.16 (**B.-Farina**). *Supposons* $2 \le N \le 4$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution stable de* (SPE+) *où* $f \in C^1([0,+\infty))$ *est une fonction positive et croissante avec* $f \not\equiv 0$. *Si*

$$u(x) = O(|x|^{\frac{4-N}{2}} \ln^{1/2}|x|) \qquad \textit{quand} \quad |x| \to +\infty, \tag{0.8.13}$$

alors, $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f(0) = f'(0) = 0$.

Les deux résultats suivants traitent de solutions stables en dehors d'un compact de $\mathbb{R}^N_+$.

Théorème 0.8.17 (**B.-Farina**). *Supposons* $2 \le N \le 9$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution de* (SPE+) *qui est stable en dehors d'un compact de* $\mathbb{R}^N_+$. *Soit* $f \in C^1([0,+\infty))$ *une fonction positive et convexe avec* $f(0) = 0$ *et* $f \not\equiv 0$.
Alors, $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f'(0) = 0$.

Théorème 0.8.18 (**B.-Farina**). *Supposons* $2 \le N \le 10$ *et soit* $u \in C^2(\overline{\mathbb{R}^N_+})$ *une solution de* (SPE+) *qui est stable en dehors d'un compact de* $\mathbb{R}^N_+$. *Soit* $f \in C^1([0,+\infty))$ *une fonction positive et croissante satisfaisant*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty, \qquad \limsup_{t\to+\infty} \frac{f(t)}{t^\theta} < +\infty. \tag{0.8.14}$$

pour un $\theta > 1$.
Alors, $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f(0) = f'(0) = 0$.

Dans le cas des solutions monotones, la flexibilité de notre approche nous permet d'étendre certains des résultats précédents au cas des inégalités différentielles, même sans imposer de conditions aux limites. Plus précisément, nous avons

Théorème 0.8.19 (**B.-Farina**). *Supposons* $N = 2, 3$ *et soit* $u \in C^2(\mathbb{R}^N_+)$ *une solution de*

$$\begin{cases} -\Delta u \ge f(u) & \textit{dans} \quad \mathbb{R}^N_+, \\ u \ge 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \ge 0 & \textit{dans} \quad \mathbb{R}^N_+, \end{cases} \tag{0.8.15}$$

où $f \in C^0([0,+\infty))$ *satisfait* $f(t) > 0$ *pour tout* $t > 0$ *et*

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > 0.$$

Alors, $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *et* $f(0) = 0$.

En particulier, pour $N = 2, 3$ et pour tout $p \geq 1$, la seule solution positive et monotone de $-\Delta u \geq u^p$ dans $\mathbb{R}^N_+$ est la fonction identiquement nulle.

Nous notons également que le résultat ci-dessus est optimal. En effet, la fonction $u(x', x_N) = c(1 + |x'|^2)^{-\frac{1}{p-1}}$ satisfait $-\Delta u \geq u^p$ et $\frac{\partial u}{\partial x_N} \equiv 0$ dans $\mathbb{R}^N_+$, quand $N \geq 4$, $p > \frac{N-1}{N-3}$ et $c > 0$ est assez petit. Néanmoins, nous avons le résultat optimal de type Liouville suivant:

Théorème 0.8.20 (B.-Farina). *Supposons $N \geq 2$, soit $p \geq 1$ un réel et soit $u \in C^2(\mathbb{R}^N_+)$ une solution de*

$$\begin{cases} -\Delta u \geq u^p & \textit{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \end{cases} \tag{0.8.16}$$

Si $1 \leq p \leq \bar{p}(N)$, où

$$\bar{p}(N) := \begin{cases} +\infty & \textit{si} \quad N = 2, 3, \\ \frac{N-1}{N-3} & \textit{si} \quad N \geq 4, \end{cases} \tag{0.8.17}$$

alors, $u \equiv 0$ dans $\mathbb{R}^N_+$.

Si l'on se limite aux solutions de l'équation de Poisson semi linéaire sur $\mathbb{R}^N_+$, sans toutefois imposer de conditions aux limites, les résultats de classification ci-dessus peuvent être étendus au delà de la dimension $N = 4$. C'est le contenu des deux théorèmes suivants.

Théorème 0.8.21 (B.-Farina). *Supposons $2 \leq N \leq 9$ et soit $u \in C^2(\mathbb{R}^N_+)$ une solution de*

$$\begin{cases} -\Delta u = f(u) & \textit{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \end{cases} \tag{0.8.18}$$

où $f \in C^1([0, +\infty))$ satisfaisant $f(t) > 0$ pour tout $t > 0$ et

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{0.8.19}$$

Alors, soit $u \equiv 0$ in $\mathbb{R}^N_+$, ou il existe une fonction $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ tel que:

$$u(x) = u_0(x_1, ..., x_{N-1}) \qquad \textit{pour tout } x \in \mathbb{R}^N_+.$$

Si on suppose, en plus, la condition (0.8.11)*, la conclusion précédente reste vraie même en dimension $N = 10$.*

Dans le cas de l'équation de Lane-Emden $-\Delta u = u^p$, le résultat ci-dessus peut être amélioré comme suit:

Theorem 0.8.1 (B.-Farina). *Supposons $N \geq 2$ et soit $u \in C^2(\mathbb{R}^N_+)$ une solution de*

$$\begin{cases} -\Delta u = u^p & \textit{dans} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{dans} \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \textit{dans} \quad \mathbb{R}^N_+. \end{cases} \tag{0.8.20}$$

Si $N = 2, 3$, alors $u \equiv 0$ dans $\mathbb{R}^N_+$; tandis que pour $N \geq 4$, on a *(i) si* $1 \le p < \frac{N+1}{N-3}$*, alors* $u \equiv 0$ *dans* $\mathbb{R}^N_+$*;*

(ii) si $p = \frac{N+1}{N-3}$*, alors soit* $u \equiv 0$ *dans* $\mathbb{R}^N_+$ *ou*

$$u(x) = (a + b|x' - x'_0|^2)^{-\frac{N-3}{2}} \qquad \text{pour tout } x = (x', x_N) \in \mathbb{R}^N_+, \tag{0.8.21}$$

pour $a, b > 0$ *satisfaisant* $1 = ab(N-1)(N-3)$ *et pour tout* $x'_0 \in \mathbb{R}^{N-1}$*;*

(iii) si $\frac{N+1}{N-3} < p < p_{JL}(N-1)$*, où* $p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)^+}$ *est l'exposant de stabilité de Joseph-Lundgren.*

Alors, soit $u \equiv 0$ *dans* $\mathbb{R}^N_+$*, ou il existe une fonction* $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ *tel que,*

$$u(x) = u_0(x') \qquad \text{pour tout } x = (x', x_N) \in \mathbb{R}^N_+,$$

Lorsque $N \ge 4$, pour tout $\frac{N+1}{N-3} < p < p_{JL}(N-1)$, il existe une solution strictement positive de (0.8.20) dépendant seulement des $N-1$ premières variables.[22] Notons également que les solutions des points *(ii)* et *(iii)* sont instables (voir le théorème 5 et la remarque 4 de [Far07]). Finalement, on peut observer que, pour $N \le 11$, le résultat précédent donne une classification complète des solutions de (0.8.20) pour tout $p \ge 1$.

Dans la suite de ce manuscript, le lecteur pourra retrouver les chapitres 1, 2 et 3 qui contiennent respectivement les articles [BFS25], [BF25b] et [BF25a]. Le lecteur pourra donc retrouver l'ensemble des théorèmes énoncés dans cette introduction ainsi que de nouveaux résultats et leurs preuves détaillées.

[22] Pour tout $N \ge 3$ et $p > \frac{N+2}{N-2}$ les solutions radiales et strictement positives de $-\Delta u = u^p$ dans $\mathbb{R}^N$ existe toujours (voir [GS81b] ou [JL73]). Si u_0 est une telle fonction en dimension $N-1$, alors la fonction $u(x) = u_0(x')$, $x = (x', x_N) \in \mathbb{R}^N_+$, donne l'exemple désiré.

Chapter 1

Monotonicity for solutions to semilinear problems in epigraphs

Contents

1.1 Introduction and main results

We consider solutions, possibly unbounded, to the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where f is a (locally or globally) Lipschitz-continuous function satisfying

$$f(0) \geq 0,$$

and Ω is an epigraph of $\mathbb{R}^N$, with $N \geq 2$, i.e.

$$\Omega \,:=\, \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x_N > g(x')\}\,,$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a continuous function bounded from below.

The main results of the present paper prove that the solution u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω. Our monotonicity results are new. They cover the case of uniformly continuous epigraphs (not necessarily locally Lipschitz-continuous), coercive epigraphs, as well as that of a large family of merely continuous epigraphs (with possibly arbitrary growth at infinity and not necessarily coercive). Furthermore, we do not assume that u is bounded.
Half-spaces and coercive epigraphs are a very special case of the geometries covered by our analysis. Thus, our results recover, extend and complete existing ones, which mainly deal with the case of a half-space ([BCN93]-[BCN97b],[SD10],[Dan92],[Far20],[FS16],[FS17],[FV10], [GMS24],[QS06]) or a locally Lipschitz-continuous coercive epigraph ([BCN97b],[EL82],[FS16]).

Our results apply to classical solutions of (1.1.1) as well as to those in the sense of distributions. To deal with the latter case and settle the general framework in which we work, we introduce the following functional space :

$$H^1_{loc}(\overline{\Omega}) = \{u : \Omega \mapsto \mathbb{R},\ u \text{ Lebesgue-mesurable } :\ u \in H^1(\Omega \cap B(0,R)) \quad \forall\, R > 0\}.^1 \tag{1.1.2}$$

The choice of the distributional framework is quite natural and justified by the fact that, in many cases, the epigraphs we consider are merely continuous (i.e., the functions g that define them are assumed to be continuous and nothing more).

Let us first state the main results for uniformly continuous epigraphs.

Theorem 1.1.1. *Let Ω be a uniformly continuous epigraph bounded from below. Assume $f \in Lip_{loc}([0, +\infty))$ with*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0 \tag{1.1.3}$$

and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) *which bounded and uniformly continuous on finite strips.*[2]
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.[3]
If f is globally Lipschitz-continuous, the same conclusion holds true only under the sole assumption that u is uniformly continuous on finite strips.

The epigraphs considered in the previous result could be very wild, as shown by the following two-dimensional examples constructed by using the Weierstrass-type functions $g_{b,\alpha}(x) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n \pi x)$, where $b > 1$ is an integer and $\alpha \in (0,1)$. The function $g_{b,\alpha}$ is uniformly continuous, bounded and nowhere differentiable ([Har16]).

If we further assume that the epigraph satisfies a weak regularity assumption, we can prove the monotonicity result for any (locally or globally) Lipschitz-continuous function f

[1] i.e., u is Lebesgue-measurable on Ω and $u \in H^1(U)$ for any open bounded set $U \subset \Omega$.
[2] i.e., for any $R > 0$, u is bounded and uniformly continuous on the strip $\Omega \cap \{x_N < R\}$.
[3] Continuous distributional solutions of $-\Delta u = f(u)$ in Ω belong to $C^2(\Omega)$, by standard elliptic theory.

satisfying $f(0) \geq 0$ (and this by also weakening the assumptions on u).

Theorem 1.1.2. *Let Ω be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition.*
Assume $f \in Lip_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) which is bounded on finite strips.[4]
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Theorem 1.1.3. *Let Ω be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition.*
Assume $f \in Lip([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) with at most exponential growth on finite strips.[5]
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Note that Theorem 1.1.2 and Theorem 1.1.3 cover the case of uniformly continuous epigraphs bounded from below, which are not necessarily locally Lipschitz-continuous. See Example 1.7.1 in Section 1.7.

Although the following results are special cases of the preceding theorems, even in this weaker form, they are new.

Corollary 1.1.1. *Let Ω be a uniformly continuous epigraph bounded from below. Assume $f \in Lip_{loc}([0,+\infty))$ with*

$$\liminf_{t\to 0^+} \frac{f(t)}{t} > 0 \tag{1.1.5}$$

and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of (1.1.1) such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Corollary 1.1.2. *Let Ω be a globally Lipschitz-continuous epigraph bounded from below. Assume $f \in Lip_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of (1.1.1) such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Corollary 1.1.3. *Assume $\alpha \in (0,1)$ and let Ω be a globally Lipschitz-continuous epigraph bounded from below with $g \in C^{1,\alpha}_{loc}(\mathbb{R}^{N-1})$. Assume $f \in Lip_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of (1.1.1) which is bounded on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

When the epigraph is coercive, the monotonicity result holds under the sole assumption of continuity of g, i.e., we do not need to require either uniform continuity or the uniform

[4] i.e., for any $R > 0$,

$$\sup_{\Omega \cap \{x_N < R\}} u < +\infty. \tag{1.1.4}$$

[5] i.e., for any $R > 0$, there are positive numbers $A = A(R), B = B(R)$ such that

$$u(x) \leq A e^{B|x|} \qquad \forall\, x \in \Omega \cap \{x_N < R\}\,.$$

exterior cone condition for the epigraph. Moreover, this result holds regardless of the value of $f(0)$. More precisely, we have

Theorem 1.1.4. *Let Ω be a coercive continuous epigraph. Assume $f \in Lip_{loc}([0,+\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (1.1.1). *Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

To prove our main results, we first establish some new comparison principles for semilinear problems on general unbounded open sets of $\mathbb{R}^N$, and then we use them to start and to complete a modified version of the moving plane method adapted to the geometry of the epigraph Ω. Then, by a delicate analysis based on the translation invariance of the equation and on some fine compactness and regularity arguments, we obtain the desired results of monotonicity.

The flexibility of our methods also allows to prove various extensions of our main results to a large class of merely continuous epigraphs bounded from below (see the class $\mathcal{G}$ introduced in definition 2.2.2 and the list of examples that follows it). Those general results (see Theorems 1.5.1, 1.5.2 and 1.5.3) are stated, discussed and proven in Section 1.5. Below we illustrate them with some particularly evocative examples.

Theorem 1.1.5. *Let $N \geq 2$ and let $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ be a continuous function such that :*

1. *$N = 2$ and $\lim_{x \mapsto -\infty} g(x) \in (-\infty, +\infty]$, $\lim_{x \mapsto +\infty} g(x) \in (-\infty, +\infty]$;*
2. *$N = 2$ and g is quasiconvex (resp. quasiconcave) and bounded from below (in particular monotone or convex functions bounded from below qualify);*
3. *$N \geq 2$ and g of class C^2, bounded from below and with bounded second derivatives;*
4. *$N \geq 2$ and g bounded from below, $g = \varphi \circ \theta$, where θ is uniformly continuous on $\mathbb{R}^{N-1}$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is a continuous bijection;*
5. *$g(x_1, \dots, x_{N-1}) = g(x_1, \dots, x_{n-1})$, where $2 \leq n < N$ and $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is one of the functions defined in one of the items* $(1) - (4)$.

Let Ω be the epigraph defined by g and assume $f \in Lip_{loc}([0,+\infty))$ with

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0. \tag{1.1.6}$$

(i) If $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ is a distributional solution to (1.1.1) *which is bounded and uniformly continuous on finite strips.*
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.
If f is globally Lipschitz-continuous, the same conclusion holds true only under the sole assumption that u is uniformly continuous on finite strips.

(ii) If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical solution of (1.1.1) *such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Typical examples of functions g, for which Theorem 1.1.5 applies but not Theorem 1.1.1 (resp. Corollary 1.1.1), are provided by : $g_1(x_1) = e^{x_1} - 4\arctan(x_1) - 2$, $g_2(x_1) = e^{e^{x_1}}$

if $N = 2$ and $g(x_1, \ldots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1} \sin(jx_j)$, $g(x_1, \ldots, x_{N-1}) = e^{x_1 + \sum_{j=2}^{N-1} \cos^j(x_j)}$ if $N \geq 3$.

Theorem 1.1.6. *Let $N \geq 2$ and let Ω be as in the statement of Theorem 1.1.5. Also suppose that Ω satisfies a uniform exterior cone condition.*

(i) Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) *with at most exponential growth on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

(ii) Assume $f \in Lip_{loc}([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) *which is bounded on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Notice that Theorem 1.1.6 applies (for instance) to epigraphs defined by $g(x_1, \ldots, x_N) = e^{e^{x_1}}$ or $g(x_1, \ldots, x_N) = e^{x_1} - 4\arctan(x_1) - 2$ if $N \geq 1$ and to $g(x_1, \ldots, x_N) = x_1^4 + e^{x_2}$ if $N \geq 2$.

The next result applies to solutions of (1.1.1) where Ω is merely a continuous epigraph and f is a non-increasing function, possibly discontinuous, and with no restriction on the sign of $f(0)$ (see Theorem 1.5.3 in Section 5).

Theorem 1.1.7. *Let Ω be a continuous epigraph bounded from below and let $f : [0, \infty) \mapsto \mathbb{R}$ be any non-increasing function. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (1.1.1) *with subexponential growth on finite strips* [6].
Then u is non-decreasing, i.e., $\frac{\partial u}{\partial x_N} \geq 0$ in Ω.[7]
Moreover, if $f \in Lip_{loc}$, then u is strictly increasing, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

We conclude the introduction by observing that

1) the new comparisons principles we have demonstrated have also allowed us to prove some new results of uniqueness and symmetry for solutions (possibly unbounded and sign-changing) to the homogeneous Dirichlet BVP for the semilinear Poisson equation in fairly general unbounded domains (see Corollary 1.2.1 and Corollary 1.2.2 in Section 1.2);

2) as an application of our new monotonicity theorems, we also prove some new classification and/or non-existence results for solutions to the nonlinear problem (1.1.1) (see Section 1.6).

Finally we mention some interesting recent results contained in ([BG22],[BG25a],[BG25b]) where the authors study the monotonicity and uniqueness of bounded positive solutions to the Dirichlet problem of $-\Delta u = f(u)$, for the special class of bistable or nonlinear-field type nonlinearities. Those monotonicity and uniqueness results are established for a class

[6]i.e., for any $R > 0$,

$$\limsup_{\substack{|x| \to \infty, \\ x \in \Omega \cap \{x_N < R\}}} \frac{\ln u(x)}{|x|} \leq 0.$$

[7] Note that $u \in C^1(\Omega)$, since $f(u) \in L^\infty_{loc}(\Omega)$.

of uniformly Lipschitz unbounded domains which are bounded from below. Furthermore the assumptions on the unbounded domains are shown to be sharp.
The method developed in [BG22],[BG25a],[BG25b] makes use of spectral information (stability) and therefore differs from the approach we have adopted in our work.

1.2 Some new comparison principles on unbounded domains and their applications to the uniqueness and symmetry of solutions to the semilinear Poisson equation

In this section we prove some new comparison principles for solutions of semilinear problems on unbounded open sets, whose section has some "good properties". Those results recover and considerably improve the comparison principle on strips (or on open subsets included in strips) proved by the second author in [Far20]. Their role in this paper is twofold:

1) they are crucial for proving our monotonicity results on epigraphs,

2) they allow us to prove some new results of uniqueness and symmetry for solutions (possibly unbounded and sign-changing) to the homogeneous Dirichlet BVP for the semilinear Poisson equation in fairly general unbounded domains.

To this end we need the following

Definition 1.2.1. Assume $N \geq 2$. Let Ω be an open subset of $\mathbb{R}^N$ and let ν be a unit vector of $\mathbb{R}^N$.
We denote by $\mathbb{R}\nu$ the vector space spanned by the unit vector ν and by $\{\nu\}^\perp$ the orthogonal complement of ν in $\mathbb{R}^N$.

(i) We shall say that Ω is `locally bounded in the direction` ν if

$$\forall R > 0 \qquad C_\nu(R) = (B'(0', R) \times \mathbb{R}\nu) \cap \Omega \quad \text{is a bounded subset of } \mathbb{R}^N. \tag{1.2.1}$$

Here $B'(0', R)$ denotes the $N-1$-dimensional open ball of radius R centered at the origin $0'$ of $\{\nu\}^\perp$.

(ii) For every $x' \in \{\nu\}^\perp$ let us define the set $S^\nu_{x'} := (\{x'\} \times \mathbb{R}\nu) \cap \Omega$. We shall say that Ω has `bounded section in the direction` ν if

$$\mathtt{S}_\nu(\Omega) := \sup_{x' \in \{\nu\}^\perp} \mathcal{L}^1(S^\nu_{x'}) < +\infty, \tag{1.2.2}$$

where $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure.

The positive number $\mathtt{S}_\nu(\Omega)$ will be called the `section of` Ω `in the direction` ν.

(iii) We shall say that Ω has `good section in the direction` ν if it is locally bounded in the direction ν and if it also has bounded section in the direction ν (that is, if it satisfies both (1.2.1) and (1.2.2)).

The following remark shows how large the families of sets defined above are.

Remark 1.2.1. (i) Open sets (possibly unbounded and not necessarily connected) that are bounded in a fixed direction ν (i.e., open sets such that, up to a rotation, are included in a finite strip $\{x = (x', x_N) \in \mathbb{R}^N : \alpha < x_N < \beta\}$, with $\alpha, \beta \in \mathbb{R}$) have good section in the direction ν. Indeed, they clearly satisfy (1.2.1) as well as (1.2.2) with $\mathtt{S}_\nu(\Omega) \leq \beta - \alpha$.

(ii) The family of open sets with good section in a fixed direction ν is much larger than the one of sets that are bounded in the direction ν. Indeed, the unbounded open connected set $\Omega_1 = \{(x, y) \in \mathbb{R}^2 : |x| - h(x) < y < |x| + h(x)\} \cup \{(x, y) \in \mathbb{R}^2 : -|x| - h(x) < y < -|x| + h(x)\} \cup \{(x, y) \in \mathbb{R}^2, |y| < 1\}$, where $h(x) = sinh^{-1}(e^{-|x|})$,(see Figure 1.1) has good section in the direction $e_2 = (0, 1)$, with $\mathtt{S}_{e_2}(\Omega_1) \leq 4$, but it is unbounded in any direction. Actually, it is not contained in any affine half-plane. Also note that the Lebesgue measure of Ω_1 is infinite.

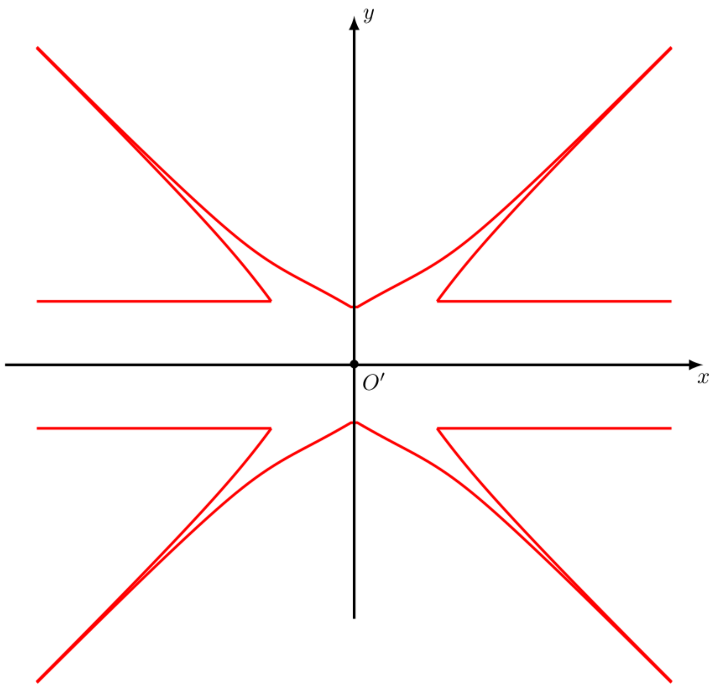

Figure 1.1: Ω_1

(iii) We also note that open sets with bounded section in a fixed direction ν are not necessarily locally bounded in the direction ν and vice versa. Consider the open sets $\Omega_2 = \bigcup_{n \geq 1} \left\{(x, y) \in \mathbb{R}^2 : n < y < n + \frac{1}{2^n}\right\}$ and $\Omega_3 = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^2\}$, then Ω_2 has bounded section in the direction e_2, but it is not locally bounded in that direction, while Ω_3 is locally bounded in the direction e_2, but it has not bounded section in the direction e_2.

Our first comparison principle is the content of the next result.

Theorem 1.2.1. *Assume $N \geq 2$. Let Ω be an open subset of $\mathbb{R}^N$ and let ν be a unit vector of $\mathbb{R}^N$.*

(i) Assume that Ω has good section in the direction ν. Let $f \in Lip_{loc}(\mathbb{R})$, $M > 0$ and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases} \tag{1.2.3}$$

Then, there exists $\varepsilon = \varepsilon(f, M) > 0$ such that

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \text{ in } \Omega. \tag{1.2.4}$$

(ii) Assume that Ω has good section in the direction ν. Let $f \in C^0(\mathbb{R})$ be a non-increasing function, $M > 0$ and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases} \tag{1.2.5}$$

Then, $u \leq v$ in Ω.

(iii) Assume that Ω has bounded section in the direction ν. Let $f \in Lip_{loc}(\mathbb{R})$, $M > 0$ and $u, v \in Lip_{loc}(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in & \mathcal{D}'(\Omega), \\ |u|, |v| \leq M & in & \Omega, \\ |\nabla u|, |\nabla v| \leq M & a.e.\ in & \Omega, \\ u \leq v & on & \partial\Omega. \end{cases} \tag{1.2.6}$$

Then, there exists $\varepsilon = \varepsilon(f, M) > 0$ such that

$$\mathtt{S}_\nu(\Omega) < \varepsilon \implies u \leq v \text{ in } \Omega. \tag{1.2.7}$$

More generally, we have the following results[8]

Theorem 1.2.2. *Assume $\gamma \geq 0$, $\delta \geq 0$, $N \geq 2$ and let Ω be an open subset of $\mathbb{R}^N$ with good section in the direction e_N, the last vector of the canonical base of $\mathbb{R}^N$, such that*

$$\sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma |x_N|} dx_N \right) < +\infty. \tag{1.2.8}$$

Let $f = f_1 + f_2$, with $f_1 \in Lip(\mathbb{R})$ and $f_2 : \mathbb{R} \mapsto \mathbb{R}$ be a non-increasing function.

Let $a > 0$ and $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & in \quad \mathcal{D}'(\Omega), \\ |u|, |v| \leq a|x|^\delta e^{\gamma|x|} & in \quad \Omega, \\ u \leq v & on \quad \partial\Omega. \end{cases}$$

[8] For sake of clarity, Theorem 1.2.2 and Theorem 1.2.3 are stated with respect to the direction e_N, the last vector of the canonical base of $\mathbb{R}^N$. Since the considered problems are invariant by rotation, it is clear that they also hold true if the unit vector e_N is replaced by any unit vector ν (and with the corresponding natural modification of condition (1.2.8) in the case of Theorem 1.2.2).

Then, there exists $\varepsilon = \varepsilon(L_{f_1}, \gamma) > 0$ *such that*

$$\mathtt{S}_{e_N}(\Omega) < \varepsilon \implies u \le v \text{ in } \Omega.$$

Here, L_{f_1} *denotes the Lipschitz constant of* f_1.

Some remarks are in order

Remark 1.2.2. (i) Any open set Ω included in a strip $\{x = (x', x_N) \in \mathbb{R}^N : \alpha < x_N < \beta\}$, with $\alpha, \beta \in \mathbb{R}$, clearly satisfies the assumption (1.2.8) for every $\gamma, \delta \ge 0$.

(ii) Notice that the preceding theorem applies to the set Ω_1 described in Remark 1.2.1. Indeed, Ω_1 satisfies the assumption (1.2.8) for any $\delta \ge 0$ and any $\gamma \in [0, 1/2]$.

(iii) The previous comparison result applies, in particular, to the case where f is globally Lipschitz-continuous. We shall use it in this form to prove Theorem 1.1.3.

When f is non-increasing on $\mathbb{R}$, the comparison principle holds even without the smallness assumption on the section of Ω. More precisely we have the following

Theorem 1.2.3. *Assume* $N \ge 2$ *and let* Ω *be an open subset of* $\mathbb{R}^N$ *bounded in the direction* e_N*, the last vector of the canonical base of* $\mathbb{R}^N$*. Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be a non-increasing function and* $u, v \in H^1_{loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$ *such that*

$$\begin{cases} -\Delta u - f(u) \le -\Delta v - f(v) & \text{in} \quad \mathcal{D}'(\Omega), \\ |u|, |v| \le a|x|^{\delta} e^{\gamma|x|} & \text{in} \quad \Omega, \\ u \le v & \text{on} \quad \partial\Omega, \end{cases}$$

for some $a > 0$, $\delta \ge 0$ *and* $\gamma \in \left[0, \frac{\pi}{4\mathtt{S}_{e_N}(\Omega)\sqrt{e-1}}\right)$.
Then, $u \le v$ *in* Ω.

The smallness assumption on the section of Ω is necessary for the validity of both Theorem 1.2.1 (item (i) and item (iii)) and Theorem 1.2.2. Indeed, the functions $u = \sin(y)$ and $v \equiv 0$ satisfy $-\Delta u - u = 0 = -\Delta v - v$ on the two-dimensional strip $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\}$ and $u \le v$ on $\partial\Omega$, but the conclusion of the comparison principles fails.

That the growth assumption on u in Theorem 1.2.3 is necessary to ensure the comparison principle is seen from the following classical example involving harmonic functions on finite strips of $\mathbb{R}^2$, namely, when $f \equiv 0$. It is well-known that, for any integer $k \ge 1$, the function $u_k(x, y) = \sum_{m=1}^{k} (e^{mx} + e^{-mx}) \sin(mx)$ is harmonic on the strip $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < \pi\}$ and vanishes on $\partial\Omega$. Therefore, a restriction on the exponential growth must be prescribed in order to get the desired results.

As an immediate consequence of our comparison principles, we have the following result about the uniqueness of the homogeneous Dirichlet problem.

Corollary 1.2.1. *Let Ω and f be as in the statement of Theorem 1.2.1 (resp. Theorem 1.2.2 or Theorem 1.2.3). Then, the Dirichlet problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega), \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.9}$$

has, at most, one solution u satisfying the regularity and the growth conditions stated in Theorem 1.2.1 (resp. Theorem 1.2.2 or Theorem 1.2.3).
In particular, if $f(0) = 0$, the function $u \equiv 0$ is the only solution to the Dirichlet problem (1.2.9).

The examples considered after Theorem 1.2.3, prove that the preceding result is no longer true if either the assumptions on the size of Ω or those on the growth of u are not met.

An immediate consequence of Corollary 1.2.1 is the following general symmetry result for solutions to the homogeneous Dirichlet problem (1.2.9).

Corollary 1.2.2. *Let Ω, f be as in the statement of Corollary 1.2.1 and assume that the Dirichlet problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathcal{D}'(\Omega), \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.10}$$

has a solution u satisfying the regularity and the growth conditions stated in Corollary 1.2.1.

Let ρ be an isometry of $\mathbb{R}^N$, $N \geq 2$. If Ω is invariant with respect to ρ, i.e., Ω satisfies $\rho(\Omega) = \Omega$, then the solution u inherits the same symmetry, namely, $u(x) = u(\rho(x))$ for any $x \in \Omega$.

Let us illustrate the above result on the *torsion problem* for an infinite solid bar. Let us start with the case of an infinite solid straight bar with spherical cross section. That is, the study of the solutions to the problem

$$\begin{cases} -\Delta u = K & \text{in } \mathcal{D}'(\Omega), \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.11}$$

where $\Omega = \Omega_{K,R} := \mathbb{R}^{N-K} \times B_R^K$, where $1 \leq K < N$ is an integer, and B_R^K denotes the open ball of $\mathbb{R}^K$ centered at the origin and of radius $R > 0$.

Since $\Omega_{K,R}$ is invariant with respect to any translation in the variables $x_1, \ldots, x_{N-K}$ and also to any rotation in the variables $x_{N-K+1}, \ldots, x_N$, Corollary 1.2.2[9] tell us that the unique solution u of (1.2.11), with subexponential growth at infinity, must be independent of $x_1, \ldots, x_{N-K}$ and radially symmetric in the variables $x_{N-K+1}, \ldots, x_N$.
It is therefore easy to check that the function

$$u(x) = \frac{R^2 - (x_{N-K+1}^2 + \ldots + x_N^2)}{2}, \qquad x \in \Omega_{K,R}$$

[9] applied with the non-increasing function $f \equiv K$.

is the only solution to the problem (1.2.11) with subexponential growth at infinity.

The previous analysis extends to general infinite solid bars (not necessarily straight). For instance, it applies to the following examples :

1) $\Omega = \mathbb{R}^{N-K} \times \omega$, where $1 \leq K < N$ is an integer and ω is an open bounded subset of $\mathbb{R}^K$. In this case, the unique solution u of (1.2.11) with subexponential growth at infinity, depends only on the variables $x_{N-K+1}, \dots, x_N$.

2) $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, -\varphi(x_1) < x_2 < \varphi(x_1)\}$, where $\varphi : \mathbb{R} \mapsto (0, +\infty)$ is a bounded continuous function. In this case the solution u must be symmetric with respect to the line $x_2 = 0$.

3) Ω is a domain of revolution of $\mathbb{R}^N$,$N \geq 3$, i.e., $\Omega = \left\{ x \in \mathbb{R}^N \,:\, \sqrt{x_2^2 + \ldots + x_N^2} < \varphi(x_1) \right\}$, where $\varphi : \mathbb{R} \mapsto (0, +\infty)$ is a bounded continuous function. Here, u must have the form $u = u(x_1, \sqrt{x_2^2 + \ldots + x_N^2})$.

Also observe that, if φ is T-periodic, with $T > 0$, then the solution u is T-periodic in the x_1 variable.

It is clear that the above discussions and results also hold true for general non linear function f satisfying the assumptions of Corollary 1.2.2.

Now we are ready to prove our comparison principles.

Proof of Theorem 1.2.1. Since our problem is invariant under rotation, we may and do suppose that $\nu = e_N$, the last vector of the canonical base of $\mathbb{R}^N$. By assumption, for every $\phi \in C_c^\infty(\Omega)$, $\phi \geq 0$, we have

$$\int_\Omega \nabla(u - v)\nabla\phi \leqslant \int_\Omega (f(u) - f(v))\phi. \tag{1.2.12}$$

Let us first consider the cases (i) and (ii), where we suppose that Ω has good section in the direction e_N.

Observe that, for any $\Psi \in C_c^{0,1}(\mathbb{R}^{N-1})$ and for any open ball $B' \subset \mathbb{R}^{N-1}$ with $\operatorname{supp}(\Psi) \subset B'$, we have $g := (u-v)^+\Psi^2 \in C^0(\overline{\Omega})$. Let us prove that g also belongs to $H_0^1(\Omega \cap (B' \times \mathbb{R}))$. The assumption (1.2.1) ensures that $\Omega \cap (B' \times \mathbb{R})$ is a bounded open subset of $\mathbb{R}^N$ thus, from the assumption $u, v \in H^1_{\rm loc}(\overline{\Omega}) \cap C^0(\overline{\Omega})$, it follows that $(u - v)^+$ is a bounded, continuous function on the set $\overline{\Omega \cap (B' \times \mathbb{R})}$ such that $(u - v)^+ \in H^1(\Omega \cap (B' \times \mathbb{R}))$. Also, since $\Psi^2 \in C_c^{0,1}(\mathbb{R}^{N-1})$, we have $g \in H^1(\Omega \cap (B' \times \mathbb{R}))$ and

$$\nabla g = \nabla((u - v)^+)\Psi^2 + 2\Psi(u - v)^+\nabla\Psi \quad \text{in} \quad \mathcal{D}'(\Omega \cap (B' \times \mathbb{R})). \tag{1.2.13}$$

Also note that

- if $x \in \partial\Omega \cap (B' \times \mathbb{R})$, then $g(x) = 0$, since $u(x) \leqslant v(x)$ on $\partial\Omega$,
- if $x = (x', x_N) \in \overline{\Omega} \cap \partial(B' \times \mathbb{R})$, then $\Psi(x') = 0$, and so $g(x) = 0$,

hence $g = 0$ on $\partial(\Omega\cap(B'\times\mathbb{R})) = (\partial\Omega\cap(B'\times\mathbb{R}))\cup(\overline{\Omega}\cap\partial(B'\times\mathbb{R}))$. Then $g \in H^1_0(\Omega\cap(B'\times\mathbb{R}))$, $g \geq 0$, and so we can find a sequence of functions $\phi_n \in C^\infty_{\mathrm{c}}(\Omega \cap (B' \times \mathbb{R}))$, $\phi_n \geq 0$, such that

$$\lim_{n\to+\infty} \|\phi_n - g\|_{H^1(\Omega\cap(B'\times\mathbb{R}))} = 0.$$

By (1.2.12), for any $n \geq 1$ we have

$$\int_{\Omega\cap(B'\times\mathbb{R})} \nabla(u-v)\nabla\phi_n = \int_\Omega \nabla(u-v)\nabla\phi_n \leq \int_\Omega (f(u)-f(v))\phi_n = \int_{\Omega\cap(B'\times\mathbb{R})} (f(u)-f(v))\phi_n$$

and passing to the limit, we obtain

$$\int_\Omega \nabla(u-v)\nabla g \leq \int_\Omega (f(u)-f(v))g, \tag{1.2.14}$$

since the support of g is contained in $\Omega \cap (B' \times \mathbb{R})$. Plugging (1.2.13) into (1.2.14) yields

$$\int_\Omega \nabla(u-v)(\nabla(u-v)^+\Psi^2 + (u-v)^+ 2\Psi\nabla\Psi) \leq \int_\Omega (f(u)-f(v))(u-v)^+\Psi^2 \tag{1.2.15}$$

and so

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 + \int_\Omega (u-v)^+ 2\Psi\nabla(u-v)^+\nabla\Psi \leq L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2, \tag{1.2.16}$$

where $L_{f,M}$ is any positive number greater or equal to the Lipschitz constant of f on the compact set $[-M, M]$ if we are in the case (i), while $L_{f,M} = 0$ if the case (ii) is in force. Hence

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \leq -2\int_\Omega (u-v)^+\Psi\nabla(u-v)^+\nabla\Psi + L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2$$

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \leq -2\int_\Omega (\sqrt{2}(u-v)^+\nabla\Psi)\left(\frac{\Psi\nabla(u-v)^+}{\sqrt{2}}\right) + L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2$$

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \leq 2\int_\Omega (\sqrt{2}(u-v)^+|\nabla\Psi|)\left(\frac{|\Psi||\nabla(u-v)^+|}{\sqrt{2}}\right) + L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2$$

and by Young's inequality

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \leq \int_\Omega 2((u-v)^+)^2|\nabla\Psi|^2 + \int_\Omega \frac{\Psi^2|\nabla(u-v)^+|^2}{2} + L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2.$$

Therefore we have

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \leq 4\int_\Omega ((u-v)^+)^2|\nabla\Psi|^2 + 2L_{f,M}\int_\Omega ((u-v)^+)^2\Psi^2. \tag{1.2.17}$$

Since

$$|\nabla(u-v)^+|^2 \geq |\partial_N(u-v)^+|^2 \quad \text{a.e. on} \quad \Omega,$$

we get

$$\int_\Omega |\nabla(u-v)^+|^2\Psi^2 \geq \int_\Omega |\partial_N(u-v)^+|^2\Psi^2 = \int_{B'}\int_{S^{e_N}_{x'}} |\partial_N(u-v)^+(x',x_N)|^2\Psi^2(x')dx_N dx' =$$

$$= \int_{B'} \Psi^2(x')\Big(\int_{S^{e_N}_{x'}} |\partial_N(u-v)^+(x',x_N)|^2 dx_N\Big)dx'.$$

Now we observe that, for every $x' \in B'$, the one-dimensional set $S^{e_N}_{x'}$ can be written as the disjoint union of an at most countable family of open bounded intervals (its connected components), i.e., $S^{e_N}_{x'} = \sqcup_{j \in J(x')} I_{x',j}$, with $J(x') \subset \mathbb{N}$ and $I_{x',j}$ an open bounded interval. Therefore, by Poincaré's inequality, we have

$$\begin{aligned}\int_{S^{e_N}_{x'}} |\partial_N (u-v)^+(x',x_N)|^2 dx_N &= \sum_{j \in J(x')} \int_{I_{x',j}} |\partial_N (u-v)^+(x',x_N)|^2 dx_N \geq \\ &\geq \sum_{j \in J(x')} \frac{\pi^2}{(\mathcal{L}^1(I_{x',j}))^2} \int_{I_{x',j}} ((u-v)^+(x',x_N))^2 dx_N \geq \\ &\geq \frac{\pi^2}{(\mathtt{S}_{e_N}(\Omega))^2} \sum_{j \in J(x')} \int_{I_{x',j}} ((u-v)^+(x',x_N))^2 dx_N = \\ &= \frac{\pi^2}{(\mathtt{S}_{e_N}(\Omega))^2} \int_{S^{e_N}_{x'}} ((u-v)^+(x',x_N))^2 dx_N\end{aligned}$$

and so

$$\int_\Omega |\nabla (u-v)^+|^2 \Psi^2 \geq \frac{\pi^2}{(\mathtt{S}_{e_N}(\Omega))^2} \int_\Omega ((u-v)^+)^2 \Psi^2. \tag{1.2.18}$$

Now, thanks to (1.2.17) and (1.2.18), we deduce that

$$\left(\frac{\pi^2}{(\mathtt{S}_{e_N}(\Omega))^2} - 2L_{f,M} \right) \int_\Omega ((u-v)^+)^2 \Psi^2 \leq 4 \int_\Omega ((u-v)^+)^2 |\nabla \Psi|^2. \tag{1.2.19}$$

If the case (i) is in force, we set $\varepsilon(f,M) = \frac{\pi}{\sqrt{2L_{f,M}}}$. Then, if $\mathtt{S}_{e_N}(\Omega) < \varepsilon(f,M)$ we have

$$\int_\Omega ((u-v)^+)^2 \Psi^2 \leq C_1(f,M,\Omega) \int_\Omega ((u-v)^+)^2 |\nabla \Psi|^2, \tag{1.2.20}$$

with $C_1(f,M,\Omega) = \dfrac{4}{\frac{\pi^2}{(\mathtt{S}_{e_N}(\Omega))^2} - 2L_{f,M}} > 0.$

When the case (ii) is in force, equation (1.2.19) yields

$$\int_\Omega ((u-v)^+)^2 \Psi^2 \leq C_2(\Omega) \int_\Omega ((u-v)^+)^2 |\nabla \Psi|^2 \tag{1.2.21}$$

where $C_2(\Omega) = \frac{4\mathtt{S}^2_{e_N}(\Omega)}{\pi^2}$ (and without any restriction on the size of $\mathtt{S}_{e_N}(\Omega)$).

For $0 < a < b$, set $h = b - a$ and consider the Lipschitz-continuous function

$$\begin{aligned}\eta_h :& \mathbb{R}^+ \to \mathbb{R} \\ t \mapsto& \begin{cases} 1 & \text{if} \quad t \in [0,a], \\ \frac{b-t}{h} & \text{if} \quad t \in [a,b], \\ 0 & \text{if} \quad t \in [b,+\infty). \end{cases}\end{aligned}$$

For any $x' \in \mathbb{R}^{N-1}$, set $\Psi_h(x') := \eta_h(|x'|)$, then $\Psi_h \in C^{0,1}_c(\mathbb{R}^{N-1})$ and $|\nabla' \Psi_h(x')| \leq \frac{1}{h}$ for almost every $x' \in \mathbb{R}^{N-1}$.

For $r > 0$, recall that $C_{e_N}(r) = \Omega \cap (B'(0',r) \times \mathbb{R})$ and consider the function defined by

$$w(r) := \int_{C_{e_N}(r)} ((u-v)^+)^2. \tag{1.2.22}$$

If the case (i) is in force, from (1.2.20) we have

$$\begin{aligned} w(a) = \int_{C_{e_N}(a)} ((u-v)^+(x))^2 &\le \int_\Omega ((u-v)^+(x))^2 \Psi_h^2(x') \\ &\le C_1(f,M,\Omega) \int_\Omega ((u-v)^+(x))^2 |\nabla \Psi_h(x')|^2 \\ &\le \frac{C_1(f,M,\Omega)}{h^2} \int_{C_{e_N}(b)\setminus C_{e_N}(a)} ((u-v)^+(x))^2 \end{aligned}$$

and by adding $\frac{C_1(f,M,\Omega)}{h^2} w(a)$ to both sides of the latter, we have

$$w(a) \le \frac{1}{\frac{h^2}{C_1(f,M,\Omega)} + 1} w(a+h), \qquad \forall\, a, h > 0.$$

To conclude it is enough to prove that $w \equiv 0$. If not, we can find $A > 0$ such that

$$w(A) > 0, \tag{1.2.23}$$

and so, we can apply Lemma 3.1 in [BFP24] with $\alpha = \frac{1}{C_1(f,M,\Omega)} > 0$, $\delta = 0, \gamma = 2$ and $\beta = 1$ to infer that, for any $h > 0$, the function w satisfies

$$w(R) \ge w(A)(\alpha h^2 + 1)^{\frac{R-A}{h} - 1}, \qquad \forall\, R \ge A + h. \tag{1.2.24}$$

If we take $h = \sqrt{\frac{e-1}{\alpha}} > 0$, then

$$w(R) \ge w(A)(\alpha h^2 + 1)^{\frac{R-A}{h} - 1} = w(A) e^{-(\frac{A}{h} + 1)} e^{\frac{R}{h}}, \qquad \forall\, R >> 1. \tag{1.2.25}$$

On the other hand, the boundedness of u and v leads to

$$w(R) \le 4M^2 \mathcal{L}^N(C_{e_N}(R)) \le 4M^2 \mathrm{S}_{e_N}(\Omega) \omega_{N-1} R^{N-1}, \qquad \forall\, R > 0, \tag{1.2.26}$$

where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure and ω_{N-1} is the Lebesgue measure of unit ball of $\mathbb{R}^{N-1}$. Inequality (1.2.26) is in contradiction with (1.2.25), so $w \equiv 0$ and the desired conclusion follows.

The case (ii) is proven in exactly the same way. Simply replace the constant $C_1(f,M,\Omega)$ by the constant $C_2(\Omega)$ in the previous argument. Indeed, since f is non-increasing, inequality (1.2.15) becomes

$$\int_\Omega \nabla(u-v)(\nabla(u-v)^+ \Psi^2 + (u-v)^+ 2\Psi \nabla \Psi) \le \int_\Omega (f(u) - f(v))(u-v)^+ \Psi^2 \le 0$$

and so (1.2.16) holds true with $L_{f,M} = 0$.

The proof in the case (iii) is similar to the one of the case (i). However, we must pay attention to the fact that in this case we do not assume that Ω is locally bounded in the direction e_N, and therefore we have to use a different path to prove that $g \in$

$H_0^1(\Omega \cap (B' \times \mathbb{R}))$.
To this end, we set $\omega = \Omega \cap (B' \times \mathbb{R})$ and we observe that $\mathcal{L}^N(\omega) < +\infty$. Indeed,

$$\mathcal{L}^N(\omega) = \int_\omega dx = \int_{B'} \Big(\int_{S_{x'}^{e_N}} dx_N \Big) dx' \leq \mathtt{S}_{e_N}(\Omega)\mathcal{L}^{N-1}(B') < +\infty,$$

where $\mathcal{L}^{N-1}$ denotes the $(N-1)$-dimensional Lebesgue-measure of the open ball $B' \subset \mathbb{R}^{N-1}$.

We already know that $g \in C^0(\overline{\omega})$ and $g = 0$ on $\partial\omega$. Moreover $g \in L^2(\omega)$ since

$$\int_\omega g^2(x)dx = \int_\omega ((u-v)^+(x))^2 \Psi^4(x')dx \leq 4M^2 \|\Psi\|^4_{L^\infty(\mathbb{R}^{N-1})} \mathcal{L}^N(\omega) < +\infty.$$

Also, g is locally Lipschitz-continuous in Ω, since $u, v \in Lip_{\mathrm{loc}}(\overline{\Omega})$ by assumption. Hence, formula (1.2.13) holds true and so $g \in H^1(\Omega \cap (B' \times \mathbb{R}))$, since

$$\int_\omega (\partial_j((u-v)^+)^2 \Psi^4 \leq 4M^2 \|\Psi\|^4_{L^\infty(\mathbb{R}^{N-1})} \mathcal{L}^N(\omega),$$

$$\int_\omega \Psi^2((u-v)^+)^2 (\partial_j \Psi)^2 \leq 4M^2 \|\Psi\|^2_{L^\infty(\mathbb{R}^{N-1})} \|\partial_j \Psi\|^2_{L^\infty(\mathbb{R}^{N-1})} \mathcal{L}^N(\omega).$$

Then, $g \in H_0^1(\omega)$ and so, we can proceed as in the proof of the case (i) to obtain the desired conclusion. $\square$

Proof of Theorem 1.2.2. We proceed as in the proof of the case (i) of Theorem 1.2.1 until (1.2.15). (This is possible, since in this case (1.2.14) holds even under the assumption $f = f_1 + f_2$, where f_2 is a non-increasing function, possibly discontinuous. Indeed, in this case $f(u)$ and $f(v)$ are bounded measurable functions on the bounded set $\Omega \cap (B' \times \mathbb{R})$). Then we obtain (1.2.16), where now $L_{f,M}$ is any positive number greater or equal to L_{f_1}, the Lipschitz constant of f_1. Here we have used that $(f(u) - f(v))(u-v)^+ \leq (f_1(u) - f_1(v))(u-v)^+$, since f_2 is non-increasing. After that, the same proof leads to inequality (1.2.19).

To complete the proof we set $\varepsilon(L_{f_1}, \gamma) = \dfrac{\pi}{\sqrt{16(e-1)\gamma^2 + 2L_{f,M}}}$ and we observe that (1.2.20) is still satisfied if $\mathtt{S}_{e_N}(\Omega) < \varepsilon(L_{f_1}, \gamma)$. Therefore, we can consider once again the function w defined by (1.2.22) and then follow the proof until (1.2.25), i.e.,

$$w(R) \geq w(A) e^{-(\frac{A}{h}+1)} e^{\frac{R}{h}}, \qquad \forall\, R >> 1, \tag{1.2.27}$$

where $h = \sqrt{\frac{e-1}{\alpha}}$, $A > 0$ and $w(A) > 0$. On the other hand, for $R > 1$, we also have

$$w(R) = \int_{C_{e_N}(R)} ((u-v)^+)^2 dx \leq 4a^2 \int_{C_{e_N}(R)} |x|^{2\delta} e^{2\gamma|x|} dx \tag{1.2.28}$$

$$\leq 4a^2 \int_{C_{e_N}(R)} (R + |x_N|)^{2\delta} e^{2\gamma(R+|x_N|)} dx \tag{1.2.29}$$

$$\leq 4a^2 (2R)^{2\delta} e^{2\gamma R} \int_{B'(0',R)} \left(\int_{S^{e_N}_{x'}} e^{2\gamma|x_N|} dx_N \right) dx' + \tag{1.2.30}$$

$$4a^2 2^{2\delta} e^{2\gamma R} \int_{B'(0',R)} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma|x_N|} dx_N \right) dx' \tag{1.2.31}$$

$$\leq 4a^2 (2R)^{2\delta} e^{2\gamma R} \int_{B'(0',R)} \left(\int_{S^{e_N}_{x'}} (1 + |x_N|^{2\delta}) e^{2\gamma|x_N|} dx_N \right) dx' \tag{1.2.32}$$

Now we observe that

$$\int_{S^{e_N}_{x'}} (1 + |x_N|^{2\delta}) e^{2\gamma|x_N|} dx_N = \tag{1.2.33}$$

$$\int_{S^{e_N}_{x'} \cap \{|x_N| \leq 1\}} (1 + |x_N|^{2\delta}) e^{2\gamma|x_N|} dx_N + \int_{S^{e_N}_{x'} \cap \{|x_N| > 1\}} (1 + |x_N|^{2\delta}) e^{2\gamma|x_N|} dx_N \tag{1.2.34}$$

$$\leq 2e^{2\gamma} \mathtt{s}_{e_N}(\Omega) + \int_{S^{e_N}_{x'}} 2|x_N|^{2\delta} e^{2\gamma|x_N|} dx_N \leq 2e^{2\gamma} \mathtt{s}_{e_N}(\Omega) + 2C_3 := C_4 < +\infty, \tag{1.2.35}$$

where $C_3 := \sup_{x' \in \mathbb{R}^{N-1}} \left(\int_{S^{e_N}_{x'}} |x_N|^{2\delta} e^{2\gamma|x_N|} dx_N \right) < +\infty$, by (1.2.8).

From the previous inequalities we infer that, for $R > 1$,

$$w(R) \leq 4a^2 (2R)^{2\delta} e^{2\gamma R} C_4 \mathcal{L}^{N-1}(B'(0',R)) = 4a^2 2^{2\delta} C_4 \mathcal{L}^{N-1}(B'(0',1)) R^{2\delta+N-1} e^{2\gamma R} \tag{1.2.36}$$

The latter inequality contradicts (1.2.27), since

$$2\gamma < \frac{1}{h} \Longleftrightarrow \mathtt{s}_{e_N}(\Omega) < \frac{\pi}{\sqrt{16(e-1)\gamma^2 + 2L_{f,M}}},$$

therefore $w \equiv 0$ and so $u \leq v$ on Ω. □

Proof of Theorem 1.2.3. We proceed as in the proof of the case (ii) of Theorem 1.2.2 until (1.2.21). (This is possible, since (1.2.14) holds even when f is a non-increasing function, possibly discontinuous. Indeed, in this case $f(u)$ and $f(v)$ are bounded measurable functions on the bounded set $\Omega \cap (B' \times \mathbb{R})$). Then we consider the function w defined by (1.2.22) and then follow the proof until (1.2.25), i.e.,

$$w(R) \geq w(A) e^{-(\frac{A}{h}+1)} e^{\frac{R}{h}}, \qquad \forall\, R >> 1, \tag{1.2.37}$$

where $h = \sqrt{\frac{e-1}{\alpha}}$, $A > 0$ and $w(A) > 0$. On the other hand, we also have

$$\begin{aligned} w(R) &= \int_{C_{e_N}(R)} ((u-v)^+)^2 dx \leq 4a^2 \int_{C_{e_N}(R)} |x|^{2\delta} e^{2\gamma|x|} dx \\ &\leq 4a^2 \int_{C_{e_N}(R)} (R + |x_N|)^{2\delta} e^{2\gamma(R+|x_N|)} dx \\ &\leq 4a^2 e^{2\gamma R} \int_{B'(0',R)} \left(\int_{S^{e_N}_{x'}} (R+|x_N|)^{2\delta} e^{2\gamma|x_N|} dx_N \right) dx' \\ &\leq 4a^2 \mathcal{L}^{N-1}(B'(0',1)) \mathtt{S}_{e_N}(\Omega)(1+\beta)^{2\delta} e^{2\gamma\beta} R^{N+2\delta-1} e^{2\gamma R}, \qquad \forall\, R > 1, \end{aligned}$$

where in the latter we have used that $\Omega \subseteq \{x = (x', x_N) \in \mathbb{R}^N \,:\, -\beta < x_N < \beta\}$, with $\beta > 0$, since Ω is bounded in the direction e_N.

The latter inequality contradicts (1.2.37), since $\gamma \in \left[0, \dfrac{\pi}{4\mathtt{S}_{e_N}(\Omega)\sqrt{e-1}}\right)$ implies $2\gamma < \frac{1}{h}$. $\square$

1.3 Some uniform estimates in unbounded domains

In order to prove some of our results we need to establish some uniform estimates for solutions to semilinear problems on epigraphs satisfying a uniform exterior cone condition. Let us recall that an open set $\omega \subset \mathbb{R}^N$ (not necessarily an epigraph) satisfies a *uniform exterior cone condition* if for any $x_0 \in \partial\omega$ there exists a finite right circular cone V_{x_0}, with vertex x_0, such that $\overline{\omega} \cap V_{x_0} = \{x_0\}$ and the cones V_{x_0} are all congruent to some fixed cone V. The cone V is called the *reference cone*.

Any globally Lipschitz-continuous epigraph satisfies a uniform exterior cone condition, but the converse is not true. Indeed, the epigraph defined by the function $x \mapsto e^x$ is bounded from below, it satisfies a uniform exterior cone condition (by convexity) without being uniformly continuous.

In the following we consider a (merely) continuous function $h : \mathbb{R}^{N-1} \to \mathbb{R}$, its epigraph

$$\omega \,:=\, \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x_N > h(x') \right\}$$

and, for any $R > 0$ and any $\tau > \sup_{B'(0',R)} |h|$, we set

$$\mathfrak{C}^h(0', R, \tau) = \left\{ x = (x', x_N) \in \mathbb{R}^N,\, x' \in B'(0', R) \text{ and } h(x') < x_N < \tau \right\}, \tag{1.3.1}$$

the intersection of the epigraph ω with the truncated cylinder $B'(0', R) \times (-\tau, \tau)$ and

$$\widehat{\mathfrak{C}^h}(0', R, \tau) = \mathfrak{C}^h(0', R, \tau) \cup \{x = (x', x_N) \in B'(0', R) \times \mathbb{R},\, x_N = h(x')\}\ .$$

Then, the next uniform estimate holds true.

Proposition 1.3.1. *Let ω be the epigraph of a continuous function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ and suppose that ω satisfies a uniform exterior cone condition (with reference cone V).*
Let $\widetilde{R} > 0$, $\widetilde{\tau} > \max\left\{\widetilde{R},\, 4 \sup_{\overline{B'(0',\widetilde{R})}} |h|\right\}$ and $u \in H^1(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})) \cap C^0(\overline{\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})})$ satisfy

$$\begin{cases} -\Delta u = f(u) & in \quad \mathcal{D}'(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})), \\ \quad u = 0 & on \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0', \widetilde{R}, \widetilde{\tau}), \end{cases} \tag{1.3.2}$$

where $f \in Lip_{loc}(\mathbb{R})$. Then, there exists $\alpha = \alpha(N, V) \in (0,1)$ such that $u \in C^{0,\alpha}(\overline{\mathfrak{C}^h(0', r, t)})$ for any $0 < r < \frac{\widetilde{R}}{2}$ and any $\sup_{B'(0',r)} |h| < t < \dfrac{3}{4}\widetilde{\tau}$ and

$$\|u\|_{C^{0,\alpha}(\overline{\mathfrak{C}^h(0',r,t)})} \leq C\left(L_f + |f(0)| + 1\right)\left(\|u\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))} + 1\right), \tag{1.3.3}$$

where $C = C(r, \widetilde{R}, \widetilde{\tau}, V, N) > 0$ and L_f is the Lipschitz constant of f on the interval $\left[-\|u\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))}, \|u\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))}\right]$.

In order to prove the previous proposition we need two preliminary results.

Lemma 1.3.1. *Let $N \geq 1$, $U \subset \mathbb{R}^N$ be a domain and $u \in C^2(U)$ be a solution of*

$$-\Delta u = f \quad in \quad U, \tag{1.3.4}$$

with $f \in C^0(U)$.
Let $\delta > 0$ and $y \in U$ such that $\overline{B(y,\delta)} \subset U$. Then, for any $x \in B(y, \frac{\delta}{2})$ we have

$$|u(x) - u(y)| \leq \frac{\sqrt{N}}{2^{1-\gamma}}\Big(2N\|u\|_{L^\infty(\overline{B(y,\delta)})}\delta^{-\gamma} + \|f\|_{L^\infty(\overline{B(y,\delta)})}\delta^{2-\gamma}\Big)|x-y|^\gamma, \tag{1.3.5}$$

with $\gamma \in]0,1]$.

Proof. By the mean value theorem, for any $\gamma \in]0,1]$,

$$|u(x) - u(y)| \leq \sup_{z \in \overline{B(y,\frac{\delta}{2})}} |\nabla u(z)||x-y| \leq \frac{\delta^{1-\gamma}}{2^{1-\gamma}} \sup_{z \in \overline{B(y,\frac{\delta}{2})}} |\nabla u(z)||x-y|^\gamma. \tag{1.3.6}$$

Moreover, for any $z \in \overline{B(y, \frac{\delta}{2})}$ we have $\overline{B(z, \frac{\delta}{2})} \subset \overline{B(y,\delta)}$, and so by Brandt's inequality (see [Bra69] or Theorem 3.9 in [GT01])

$$|\partial_i u(z)| \leq \frac{2N}{\delta}\|u\|_{L^\infty(\partial B(z,\frac{\delta}{2}))} + \frac{\delta}{4}\|f\|_{L^\infty(\overline{B(y,\delta)})} \qquad \forall\, i \in \{1, \cdots, N\}.$$

Thus, for any $z \in \overline{B(y, \frac{\delta}{2})}$

$$|\nabla u(z)| \leq \sqrt{N}\Big(\frac{2N}{\delta}\|u\|_{L^\infty(\overline{B(y,\delta)})} + \frac{\delta}{4}\|f\|_{L^\infty(\overline{B(y,\delta)})}\Big). \tag{1.3.7}$$

The claim then follows by combining (2.5.57) and (2.5.58). □

To prove the next result, let us recall that a domain ω satisfies a *uniform exterior regularity condition*, if

$$\exists\, \gamma > 0,\ \rho_0 > 0 \quad : \quad \forall\, \rho \in (0, \rho_0], \quad \forall\, x_0 \in \partial\omega, \qquad \frac{\mathcal{L}^N(\omega^c \cap B(x_0, \rho))}{\mathcal{L}^N(B(x_0, \rho))} \geq \gamma. \tag{1.3.8}$$

Condition (1.3.8) will be denoted by C_{γ,ρ_0}. It is well-known that any domain that satisfies a *uniform exterior cone condition*, also satisfies a *uniform exterior regularity condition* C_{γ,ρ_0}, with suitable parameters γ and ρ_0 depending only on the reference cone. Consequently, the following result applies to any epigraph that satisfies a *uniform exterior cone condition.*

Proposition 1.3.2. *Let ω be the epigraph of a continuous function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ and suppose that ω satisfies the condition C_{γ,ρ_0}.*

Let $\widetilde{R} > 0$, $\widetilde{\tau} > \max\left\{\widetilde{R}, \sup\limits_{\overline{B'(0',\widetilde{R})}} |h|\right\}$, $f \in L^N(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau}))$ and $u \in H^1(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})) \cap C^0(\overline{\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})})$ satisfy

$$\begin{cases} -\Delta u = f & \text{in} \quad \mathcal{D}'(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})), \\ \quad u = 0 & \text{on} \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0', \widetilde{R}, \widetilde{\tau}). \end{cases} \tag{1.3.9}$$

Then, there are constants $C = C(N, \rho_0, \gamma, \widetilde{R}) > 0$ and $\alpha = \alpha(N, \gamma) \in (0, 1)$ such that for any $r > 0$ and for any $x_0 \in \partial\omega \cap \left(\overline{B'(0', \widetilde{R}/2)} \times \mathbb{R}\right)$,

$$\underset{\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}) \cap B(x_0,r)}{osc} u \leq C(\|u\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))} + \|f\|_{L^N(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))}) r^\alpha.$$

Proof. To simplify the exposition we set $\Omega = \mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})$, $T = \partial\omega \cap \left(\overline{B'(0', \widetilde{R}/2)} \times \mathbb{R}\right)$, $\overline{r} = \min(\frac{\widetilde{R}}{8}, \frac{\rho_0}{2}, 1)$ and, for $r > 0$ and $x_0 \in T$, we also let $\Omega_r = \Omega \cap B(x_0, r)$.
For any $r \in (0, \overline{r}]$ we set $M_4 = \sup\limits_{\Omega_{4r}} u$, $m_4 = \inf\limits_{\Omega_{4r}} u$, $M_1 = \sup\limits_{\Omega_r} u$ and $m_1 = \inf\limits_{\Omega_r} u$ and we apply Theorem 8.26 in [GT01] to the function $W_4 = M_4 - u$, with $q = 2N$ and $p = 1$, to obtain

$$r^{-N} \|W_{4,m}^-\|_{L^1(B_{2r}(x_0))} \leq C_1(N) \left[\inf_{B_r(x_0)} W_{4,m}^- + r\|f\|_{L^N(\Omega)}\right], \tag{1.3.10}$$

where $m = \inf_{\partial\Omega \cap B_{4r}(x_0)} W_4$ (here we have used the notations of Theorem 8.26 in [GT01]). To proceed further, we observe that $m = M_4$, since $\widetilde{\tau} > \widetilde{R} \geq 4\overline{r} \geq 4r$ implies $\partial\Omega \cap B_{4r}(x_0) \subset \partial\omega \cap \widehat{\mathfrak{C}^h}(0', \widetilde{R}, \widetilde{\tau})$. Hence

$$\begin{aligned} r^{-N} \|W_{4,m}^-\|_{L^1(B_{2r}(x_0))} &\leq C_1(N) \left[\inf_{B_r(x_0)} W_{4,m}^- + r\|f\|_{L^N(\Omega)}\right] \\ &\leq C_1(N)[M_4 - M_1 + r\|f\|_{L^N(\Omega)}]. \end{aligned} \tag{1.3.11}$$

Moreover, since $2r \leq \rho_0$, we also have

$$r^{-N} \|W_{4,m}^-\|_{L^1(B_{2r}(x_0))} \geq r^{-N} M_4 \mathcal{L}^N(\Omega^c \cap B(x_0, 2r)) \geq M_4 \gamma \mathcal{L}^N(B(0, 2)). \tag{1.3.12}$$

In the latter we have used that the epigraph ω satisfies the uniform condition C_{γ,ρ_0} at x_0. Therefore, from (1.3.11) and (1.3.12) we deduce that

$$\mathcal{L}^N(B(0,2))\gamma M_4 \leq C_1[M_4 - M_1 + r\|f\|_{L^N(\Omega)}]. \tag{1.3.13}$$

The same argument applied to the function $w_4 = u - m_4$ yields

$$-\mathcal{L}^N(B(0,2))\gamma m_4 \leq C_1[m_1 - m_4 + r\|f\|_{L^N(\Omega)}], \tag{1.3.14}$$

hence, by adding (1.3.13) and (1.3.14) we obtain

$$\mathcal{L}^N(B(0,2))\gamma(M_4 - m_4) \leq C_1(M_4 - m_4) - C_1(M_1 - m_1) + 2rC_1\|f\|_{L^N(\Omega)}.$$

Thus

$$\operatorname*{osc}_{\Omega_r} u \leq \left(1 - \frac{\mathcal{L}^N(B(0,2))\gamma}{C_1}\right)\operatorname*{osc}_{\Omega_{4r}} u + 2r\|f\|_{L^N(\Omega)}. \tag{1.3.15}$$

Now, we observe that the function $w(R) := \operatorname*{osc}_{\Omega_R} u + 2R\|f\|_{L^N(\Omega)}$ is well-defined and non-decreasing on $(0,\overline{r}]$ and satisfies

$$\begin{aligned}
w(R/4) &= \operatorname*{osc}_{\Omega_{R/4}} u + \frac{R}{2}\|f\|_{L^N(\Omega)} \leq \left(1 - \frac{\mathcal{L}^N(B(0,2))\gamma}{C_1}\right)\operatorname*{osc}_{\Omega_R} u + \frac{R}{2}\|f\|_{L^N(\Omega)} + \frac{R}{2}\|f\|_{L^N(\Omega)} \\
&= \left(1 - \frac{\mathcal{L}^N(B(0,2))\gamma}{C_1}\right)\operatorname*{osc}_{\Omega_R} u + \frac{1}{2} \times 2R\|f\|_{L^N(\Omega)} \\
&\leq \max\left[1 - \frac{\mathcal{L}^N(B(0,2))\gamma}{C_1}, \frac{1}{2}\right]\left[\operatorname*{osc}_{\Omega_R} u + 2R\|f\|_{L^N(\Omega)}\right] = C_2 w(R)
\end{aligned}$$

where $C_2 = C_2(N,\gamma) := \max\left[1 - \frac{\mathcal{L}^N(B(0,2))\gamma}{C_1}, \frac{1}{2}\right] \in (0,1)$. Hence, we can apply Lemma 8.23 in [GT01] to get

$$w(R) \leq C_3\left(\frac{R}{\overline{r}}\right)^\alpha w(\overline{r}) \qquad \forall\, R \in (0,\overline{r}], \tag{1.3.16}$$

where $C_3 = C_3(N,\gamma) > 0$ and $\alpha = \alpha(N,\gamma) \in (0,1)$. From the latter we immediately infer that

$$\operatorname*{osc}_{\Omega_r} u \leq w(r) \leq C_3\left(\frac{r}{\overline{r}}\right)^\alpha w(\overline{r}) \leq 2C_3\left(\frac{r}{\overline{r}}\right)^\alpha (\|u\|_{L^\infty(\Omega)} + \|f\|_{L^N(\Omega)}) \qquad \forall\, r \in (0,\overline{r}]. \tag{1.3.17}$$

Now, if $r > \overline{r}$ then for any $x, y \in \Omega_r$

$$u(x) - u(y) \leq |u(x) - u(y)| \leq 2\|u\|_{L^\infty(\Omega)} \times 1 \leq 2\|u\|_{L^\infty(\Omega)}\left(\frac{r}{\overline{r}}\right)^\alpha,$$

thus

$$\operatorname*{osc}_{\Omega_r} u \leq 2\|u\|_{L^\infty(\Omega)}\left(\frac{r}{\overline{r}}\right)^\alpha \qquad \forall\, r > \overline{r}. \tag{1.3.18}$$

Finally, (1.3.17) and (1.3.18) imply the desired conclusion. □

We are now ready to prove Proposition 1.3.1.

Proof of Proposition 1.3.1. Set $\Omega = \mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})$, $R_1 = \min(\frac{\widetilde{R}}{2} - r, \frac{\widetilde{\tau}}{2})$ and pick $x, y \in \mathfrak{C}^h(0', r, t)$ with $x \neq y$.

1. If $|x-y| \geq \dfrac{R_1}{4}$ then, for any $\gamma \in (0,1)$, we have

$$\begin{aligned} |u(x)-u(y)| &\leq 2\|u\|_{L^\infty(\Omega)} = 2\|u\|_{L^\infty(\Omega)}\left(\frac{4}{R_1}\right)^\gamma\left(\frac{R_1}{4}\right)^\gamma \\ &\leq 2\left(\frac{4}{R_1}\right)^\gamma\|u\|_{L^\infty(\Omega)}|x-y|^\gamma = M_1\|u\|_{L^\infty(\Omega)}|x-y|^\gamma. \end{aligned} \tag{1.3.19}$$

2. If $|x-y| < \dfrac{R_1}{4}$, then we consider the set

$$T := \left\{ x = (x', x_N) \in \mathbb{R}^N \,:\, x' \in \overline{B'(0', \widetilde{R}/2)} \quad \text{and} \quad x_N = h(x') \right\}$$

and, by observing that x and y play a symmetric role, we distinguish (only) three cases.

Case 2.1(see Figure 1.2) : $d(y, T) > \frac{R_1}{2}$.

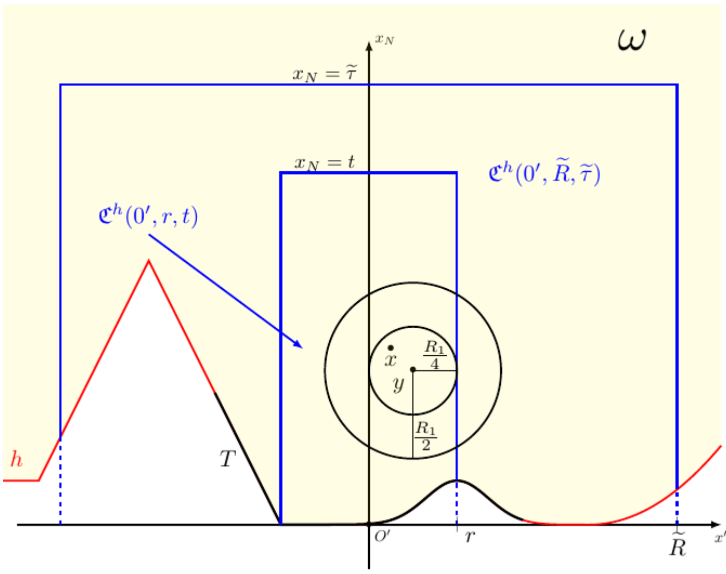

Figure 1.2: Case 2.1

In this case we have $\overline{B(y, \frac{R_1}{2})} \subset \mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})$. To see this, we first observe that $\overline{B(y, \frac{R_1}{2})} \subset \omega$, thanks to the assumption $d(y, T) > \frac{R_1}{2}$ and by the definition of R_1, and thus, for any $z \in \overline{B(y, \frac{R_1}{2})}$, we have

$$h(z') < z_N = z_N - y_N + y_N < z_N - y_N + t \leq \frac{R_1}{2} + \frac{3\widetilde{\tau}}{4} < \frac{\widetilde{\tau}}{4} + \frac{3\widetilde{\tau}}{4} = \widetilde{\tau},$$

and

$$|z'| \leq |z'-y'| + |y'| \leq \frac{R_1}{2} + r < \frac{\widetilde{R}}{2} - r + r < \widetilde{R}.$$

Now by applying Lemma 1.3.1 [10] with $\delta = \frac{R_1}{2}$ and for any $\gamma \in (0,1)$, we deduce

$$\begin{aligned} |u(x)-u(y)| &\leq \frac{\sqrt{N}}{2^{1-\gamma}}\left(2N\|u\|_{L^\infty(\Omega)}\left(\frac{R_1}{2}\right)^{-\gamma} + \|f(u)\|_{L^\infty(\Omega)}\left(\frac{R_1}{2}\right)^{2-\gamma}\right)|x-y|^\gamma \\ &\leq M_2(L_f + |f(0)| + 1)(\|u\|_{L^\infty(\Omega)} + 1)|x-y|^\gamma \end{aligned} \tag{1.3.20}$$

where $M_2 = \dfrac{\sqrt{N}}{2^{1-\gamma}}\left(2N\left(\dfrac{R_1}{2}\right)^{-\gamma} + \left(\dfrac{R_1}{2}\right)^{2-\gamma}\right)$.

Case 2.2 (see Figure 1.3) : $d(x,T) \leq d(y,T) \leq \dfrac{R_1}{2}$ and $|x-y| \geq \frac{1}{4}d(y,T)$.

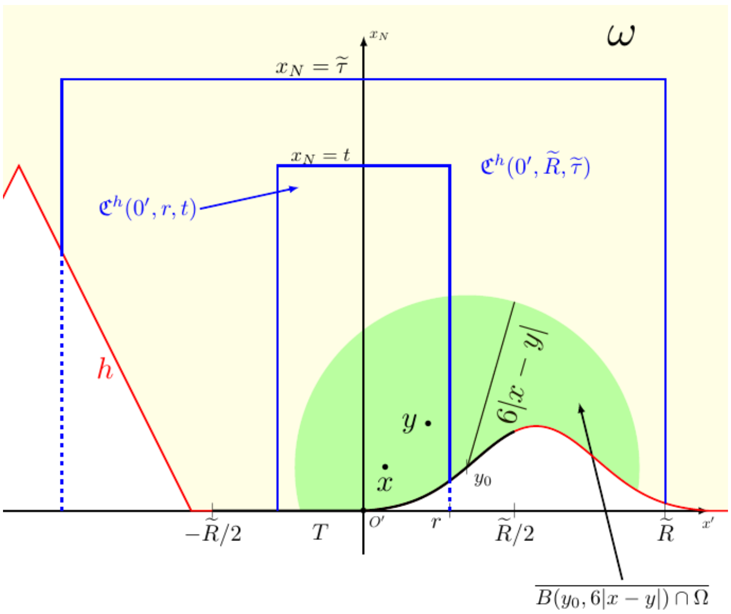

Figure 1.3: Case 2.2

Since T is a compact set, there exists $y_0 = (y_0', h(y_0')) \in T$ such that $d(y,T) = |y - y_0|$. Therefore

$$x, y \in B(y_0, 6|x-y|),$$

since

$$|x-y_0| \leq |x-y| + |y-y_0| \leq |x-y| + 4|x-y| = 5|x-y|,$$

and

$$|y-y_0| = d(y,T) \leq 4|x-y|.$$

[10] Note that $u \in C^2$ by standard elliptic regularity results, since u is continuous and f is locally Lipschitz-continuous.

Now, we apply Proposition 2.5.3 with $r = 6|x-y|$ and $x_0 = y_0$, thus there exist constants $C' = C'(N, V, \widetilde{R}) > 0$ and $\alpha = \alpha(N, V) \in (0, 1)$ such that

$$\begin{aligned} |u(x) - u(y)| &\leq 6^\alpha C'(\|u\|_{L^\infty(\Omega)} + \|f(u)\|_{L^N(\Omega)})|x-y|^\alpha \\ &\leq M_3(L_f + |f(0)| + 1)(\|u\|_{L^\infty(\Omega)} + 1)|x-y|^\alpha. \end{aligned} \tag{1.3.21}$$

where $M_3 = 6^\alpha C'\left[1 + \left(2\widetilde{\tau}\mathcal{L}^{N-1}(B'(0', 1))\widetilde{R}^{N-1}\right)^{\frac{1}{N}}\right]$.

Case 2.3 (see Figure 1.4): $d(x, T) \leq d(y, T) \leq \dfrac{R_1}{2}$ and $|x-y| < \frac{1}{4}d(y, T)$.

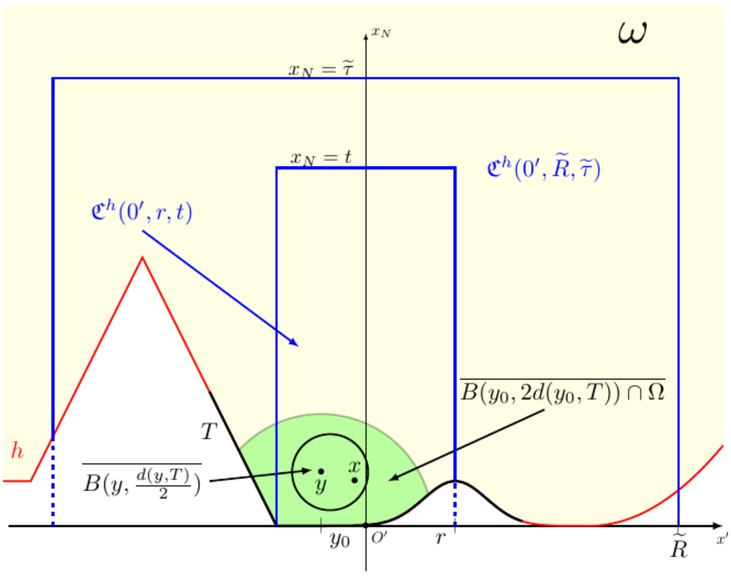

Figure 1.4: Case 2.3

By applying Lemma 1.3.1 with $\delta = \frac{d(y,T)}{2}$ and $\gamma = \alpha$,

$$\begin{aligned} |u(x) - u(y)| \leq \frac{\sqrt{N}}{2^{1-\alpha}}\Bigg(&\|f(u)\|_{L^\infty(\Omega)}\left(\frac{d(y,T)}{2}\right)^{2-\alpha} \\ &+ 2N\|u\|_{L^\infty(\overline{B(y, \frac{d(y,T)}{2})})}\left(\frac{d(y,T)}{2}\right)^{-\alpha}\Bigg)|x-y|^\alpha. \end{aligned} \tag{1.3.22}$$

Now, we want to estimate $\|u\|_{L^\infty(\overline{B(y, \frac{d(y,T)}{2})})}$. Let $z \in \overline{B(y, \frac{d(y,T)}{2})}$ and $y_0 = (y_0', h(y_0')) \in T$ such that $d(y, T) = |y_0 - y|$. Then $z \in \overline{\Omega} \cap B(y_0, 2d(y, T))$ since

$$|z - y_0| = |z - y + y - y_0| \leq |z-y| + |y - y_0| \leq \frac{3}{2}d(y, T) < 2d(y, T).$$

Thus we can apply Proposition 2.5.3 with $r = 2d(y, T)$ and $x_0 = y_0$ to get

$$|u(z)| = |u(z) - u(y_0)| \leq 2^\alpha C'(\|u\|_{L^\infty(\Omega)} + \|f(u)\|_{L^N(\Omega)})d(y, T)^\alpha.$$

Hence

$$\|u\|_{L^\infty(\overline{B(y,\frac{d(y,T)}{2})})} \leq 2^\alpha C'(\|u\|_{L^\infty(\Omega)} + \|f(u)\|_{L^N(\Omega)})d(y,T)^\alpha, \tag{1.3.23}$$

and by (2.5.24) and (1.3.23), we have

$$|u(x) - u(y)| \leq M_4(L_f + |f(0)| + 1)(\|u\|_{L^\infty(\Omega)} + 1)|x - y|^\alpha. \tag{1.3.24}$$

where $M_4 = \dfrac{\sqrt{N}}{2^{1-\alpha}} \left\{ \left(\dfrac{R_1}{4}\right)^{2-\alpha} + 2N2^{2\alpha}C' \left[1 + \left(2\tilde{\tau}\mathcal{L}^{N-1}(B'(0',1))\tilde{R}^{N-1}\right)^{\frac{1}{N}}\right]\right\}$.

The desired conclusion (2.5.13) then follows from (2.5.19),(2.5.20),(2.5.21) and (2.5.25) by taking $C = \max(M_1, M_2, M_3, M_4)$. □

1.4 Proofs

Thanks to the translation invariance of problem (1.1.1), we may and do suppose that the function $g : \mathbb{R}^{N-1} \to \mathbb{R}$, defining $\partial\Omega$, satisfies $\inf_{\mathbb{R}^{N-1}} g = 0$.

The proofs of our main results are based on the moving planes method *suitably adapted to the geometry of the epigraph* Ω. To this end, we first set the notations that will be used in our analysis.

For $0 < a < b$ and $\lambda > 0$ we set (see Figures 1.5 and 1.6):

$$\Sigma_\lambda := \{ x \in \mathbb{R}^N \ : \ 0 < x_N < \lambda \},$$
$$\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N \ : \ g(x') < x_N < b\},$$
$$\Sigma_{a,b}^g = \{x = (x', x_N) \in \mathbb{R}^N \ : \ g(x') + a < x_N < b\},$$

$$\forall x \in \Sigma_\lambda^g, \qquad u_\lambda(x) = u(x', 2\lambda - x_N).$$

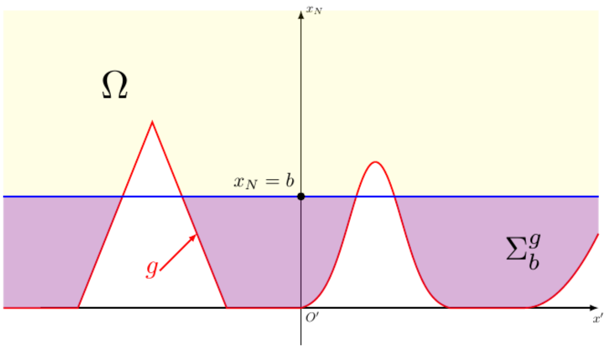

Figure 1.5: Case 2.1

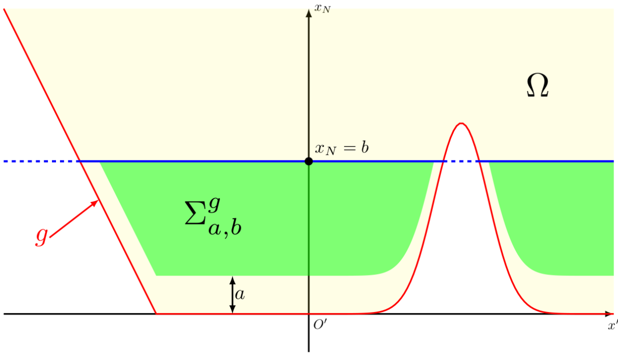

Figure 1.6: Case 2.2

In the following, for any subset $S \subseteq \mathbb{R}^N$, we denote by $UC(S)$ the set of uniformly continuous functions defined on S.

Proof of Theorem 1.1.1. We set $\Lambda := \{t > 0 \ : \ u \leqslant u_\theta \ \text{ in } \ \Sigma_\theta^g \,,\ \forall\, 0 < \theta < t\}$ and we aim at proving that

$$\tilde{t} := \sup \Lambda = +\infty.$$

To this end we split the remaining part of the proof into three steps.

Step 1 : Λ is not empty.

For any $\theta \in (0,1)$, we have $u, u_\theta \in H^1_{\mathrm{loc}}(\overline{\Sigma_\theta^g}) \cap UC(\overline{\Sigma_\theta^g})$ and

$$\begin{cases} -\Delta u - f(u) = 0 = -\Delta u_\theta - f(u_\theta) & \text{in} \quad \Sigma_\theta^g, \\ \qquad\qquad u \leq u_\theta & \text{on} \quad \partial\Sigma_\theta^g. \end{cases}$$

For the latter, notice that $\partial\Sigma_\theta^g = (\{x_N = \theta\} \cap \overline{\Omega}) \cup (\partial\Omega \cap \{x_N < \theta\})$ and so

- if $x \in \{x_N = \theta\} \cap \overline{\Omega}$, then $u(x) = u_\theta(x)$,
- if $x \in \partial\Omega \cap \{x_N < \theta\}$, then $u(x) = 0$ and $u_\theta(x) = u(x', 2\theta - x_N) > 0$, since $(x', 2\theta - x_N) \in \Omega$.

Since $\Sigma_\theta^g \subseteq \{\, x \in \mathbb{R}^N \ : \ 0 < x_N < \theta \,\}$, we also have that $\mathtt{S}_{e_N}(\Sigma_\theta^g) \leq \theta$. Therefore, when f is locally Lipschitz-continuous we can apply item (i) of Theorem 1.2.1 with any $\theta < \varepsilon(f, M)$, where $M = \sup_{\overline{\Sigma_2^g}} u$ (the latter being finite, since u is bounded on any finite strip by assumption). Therefore

$$u \leq u_\theta \quad \text{on} \quad \Sigma_\theta^g,$$

and so $(0, \varepsilon(f, M)) \subset \Lambda$.

When f is globally Lipschitz-continuous, we proceed as before by observing that u has at most linear growth on Σ_θ^g, since u is uniformly continuous on Σ_θ^g by assumption. Therefore, we can apply Theorem 1.2.2 (see Remark 1.2.2), with any $\theta < \varepsilon(L_f, \gamma)$,[11] to get

$$u \leq u_\theta \quad \text{on} \quad \Sigma_\theta^g.$$

Hence, $(0, \varepsilon(L_f, \gamma)) \subset \Lambda$.

Step 2 : $\tilde{t} = \sup \Lambda = +\infty$.

If $\tilde{t} := \sup \Lambda < +\infty$, then we have

Proposition 1.4.1. *For every $\delta \in (0, \frac{\tilde{t}}{2})$ there is $\varepsilon(\delta) > 0$ such that*

$$\forall\, \varepsilon \in (0, \varepsilon(\delta)) \qquad u \leq u_{\tilde{t}+\varepsilon} \quad on \quad \overline{\Sigma^g_{\delta, \tilde{t}-\delta}}. \tag{1.4.1}$$

Proof of Proposition 1.4.1. If the claim were not true, there would exist $\delta \in (0, \frac{\tilde{t}}{2})$ such that

$$\forall\, k \geq 1 \quad \exists\, \varepsilon_k \in \left(0, \frac{1}{k}\right), \exists\, x^k \in \overline{\Sigma^g_{\delta, \tilde{t}-\delta}} \quad : \quad u(x^k) > u_{\tilde{t}+\epsilon_k}(x^k), \tag{1.4.2}$$

[11] Here γ can be any positive real number.

and so

$$\delta \leq g((x^k)') + \delta < x_N^k < \tilde{t} - \delta, \qquad \forall k \geq 1. \tag{1.4.3}$$

Therefore, the sequence (x_N^k) is bounded, and so, up to a subsequence, we may and do suppose that $x_N^k \to x_N^\infty$, as $k \to \infty$.

Now, set

$$g_k(x') = g(x' + (x^k)') \qquad \forall x' \in \mathbb{R}^{N-1}, \quad \forall k \geq 1, \tag{1.4.4}$$

and observe that the sequence (g_k) is uniformly equicontinuous on $\mathbb{R}^{N-1}$ (since $g \in UC(\mathbb{R}^{N-1})$) and that $0 \leq g_k(0') = g((x^k)') \leq \tilde{t}$, thanks to (1.4.3). Therefore, by Ascoli-Arzelà theorem (and a standard diagonal procedure) there exists a function $g_\infty \in UC(\mathbb{R}^{N-1})$ such that, up to a subsequence, $g_k \to g_\infty$ in $C^0_{\text{loc}}(\mathbb{R}^{N-1})$.
We also observe that, passing to the limit in (1.4.3), we obtain

$$\delta \leq g_\infty(0') + \delta \leq x_N^\infty \leq \tilde{t} - \delta. \tag{1.4.5}$$

For any $k \geq 1$, let us consider $\Omega^k = \{(x', x_N) \in \mathbb{R}^N,\ x_N > g_k(x')\}$, the epigraph of g_k, $\Omega^\infty = \{(x', x_N) \in \mathbb{R}^N,\ x_N > g_\infty(x')\}$, the epigraph of g_∞ and define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \overline{\Omega}, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{1.4.6}$$

Clearly, $\tilde{u} \in UC(\{x_N < R\})$ for every $R > 0$, and so the sequence $(\tilde{u}_k)$ defined by

$$\tilde{u}_k(x) = \tilde{u}(x' + (x^k)', x_N) \qquad \forall\, x = (x', x_N) \in \mathbb{R}^N, \quad \forall k \geq 1, \tag{1.4.7}$$

is uniformly equicontinuous on $\{x_N < R\}$, for any $R > 0$. We also notice that, for k large enough, the point $(0', -1)$ belongs to $\mathbb{R}^N \setminus \overline{\Omega_k}$, since $g_k \to g_\infty$ in $C^0_{\text{loc}}(\mathbb{R}^{N-1})$ and $g_k \geq 0$ on $\mathbb{R}^{N-1}$. Hence, $\tilde{u}_k(0', -1) = 0$, for k large enough, and the sequence $(\tilde{u}_k(0', -1))$ is bounded in $\mathbb{R}$. Therefore, using once again the Ascoli-Arzelà theorem, there exists $\tilde{u}_\infty \in C^0(\mathbb{R}^N)$ such that, up to a subsequence, $\tilde{u}_k \to \tilde{u}_\infty$ in $C^0_{\text{loc}}(\mathbb{R}^N)$ and

$$\begin{cases} -\Delta \tilde{u}_\infty = f(\tilde{u}_\infty) & \text{in} \quad \mathcal{D}'(\Omega^\infty), \\ \tilde{u}_\infty \geq 0 & \text{in} \quad \Omega^\infty, \\ \tilde{u}_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \end{cases} \tag{1.4.8}$$

where the boundary condition follows by observing that,

$$\forall\, k \geq 1, \quad \forall\, x' \in \mathbb{R}^{N-1} \qquad 0 = \tilde{u}_k(x', g_k(x')),$$

and thus

$$0 = \tilde{u}_k(x', g_k(x')) \longrightarrow \tilde{u}_\infty(x', g_\infty(x')) \qquad \text{as } k \to \infty,$$

thanks to the uniform convergence of $(\tilde{u}_k)$ and (g_k) on compact sets.

By construction, $u_\infty := \tilde{u}_{\infty|\Omega^\infty} \in C^0(\overline{\Omega^\infty})$. Furthermore, $u_\infty \in C^2(\Omega^\infty)$ by (2.4.7) and standard interior regularity theory for elliptic equations, and it satisfies

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ u_\infty \geq 0 & \text{in} \quad \Omega^\infty, \\ \tilde{u}_\infty = 0 & \text{on} \quad \partial\Omega^\infty. \end{cases} \tag{1.4.9}$$

Since $u \le u_{\tilde t}$ in $\Sigma^g_{\tilde t}$, then $\tilde u_k \le \tilde u_{k,\tilde t}$ in $\Sigma^{g_k}_{\tilde t}$, for any $k \ge 1$. Passing to the limit, we get

$$u_\infty \le u_{\infty,\tilde t} \quad \text{in} \quad \overline{\Sigma^{g_\infty}_{\tilde t}}. \tag{1.4.10}$$

Also,

$$u_k(0', x^k_N) = u(x^k) > u_{\tilde t+\varepsilon_k}(x^k) = u_{k,\tilde t+\varepsilon_k}(0', x^k_N),$$

so, taking the limit as $k \to +\infty$, we have

$$u_\infty(0', x^\infty_N) \ge u_{\infty,\tilde t}(0', x^\infty_N). \tag{1.4.11}$$

In view of (2.4.4) we see that $(0', x^\infty_N) \in \overline{\Sigma^{g_\infty}_{\delta,\tilde t-\delta}} \subset \Sigma^{g_\infty}_{\tilde t}$ and so, by combining (1.4.10) and (1.4.11), we obtain

$$u_\infty(0', x^\infty_N) = u_{\infty,\tilde t}(0', x^\infty_N). \tag{1.4.12}$$

Next we use the assumption (1.1.5) to prove that $u_\infty > 0$ on Ω^∞. We first observe that, by (1.1.5), we can find $\eta > 0$ and $\varepsilon_0 > 0$ such that

$$f(t) \ge \eta t \quad \text{for any } t \in [0, \varepsilon_0], \tag{1.4.13}$$

then, we choose R large enough so that the first eigenvalue $\lambda_1(R)$ of $-\Delta$ on the open ball $B(0,R) \subset \mathbb{R}^N$ (with homogeneous Dirichlet boundary condition) satisfies $\lambda_1(R) < \eta$.

Since $g_k \to g_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^{N-1})$, there is $C(R) > 0$ such that

$$\forall k \ge 1 \qquad 0 \le g, g_k \le C(R) \qquad \text{on} \quad B'(0', 2R) \subset \mathbb{R}^{N-1}.$$

In particular, for $T > 2(C(R) + R)$, the open ball $\mathcal{B} := B((0', T), R) \subset \mathbb{R}^N$ satisfies

$$\overline{\mathcal{B}} \subset \Omega, \qquad \overline{\mathcal{B}} \subset \Omega_k \quad \forall k \ge 1, \quad \text{and} \quad \overline{\mathcal{B}} \subset \Omega_\infty \tag{1.4.14}$$

and so, also

$$\overline{\mathcal{B}} + ((x_k)', 0) \subset \Omega \quad \forall k \ge 1. \tag{1.4.15}$$

Now, set $m_0 = \min_{\overline{\mathcal{B}}} u > 0$ and denote by ϕ_1 the positive first eigenfunction of $-\Delta$ on $\mathcal{B}$ such that $\max_{\mathcal{B}} \phi_1 = 1$. Therefore, the first eigenfunction $\phi := \min\left\{\frac{m_0}{2}, \varepsilon_0\right\}\phi_1$ satisfies

$$\begin{cases} 0 < \phi < u & \text{in} \quad \mathcal{B}, \\ \Delta\phi + f(\phi) \ge 0 & \text{in} \quad \mathcal{B}, \\ \phi = 0 & \text{on} \quad \partial\mathcal{B}, \end{cases}$$

where in the latter we have used that (1.4.13) is in force.

Now, since Ω is an epigraph and (1.4.14)-(1.4.15) hold, we can use the sliding method (see for instance [BCN97b]) to get

$$\phi(x' - (x_k)', x_N) < u(x) \qquad \forall\, x \in \mathcal{B} + ((x_k)', 0) \subset \Omega,$$

that is,

$$\phi(x) < u_k(x) \quad \forall\, x \in \mathcal{B}.$$

By passing to the limit, we deduce that

$$0 < \phi(x) \le u_\infty(x) \quad \forall\, x \in \mathcal{B},$$

therefore, by (2.4.8) and the strong maximum principle, we deduce that $u_\infty > 0$ on Ω^∞.

Now we are ready to complete the proof of Proposition 1.4.1.

To this aim, we set $w_{\tilde t} := u_{\infty,\tilde t} - u_\infty$ on $\Sigma_{\tilde t}^{g_\infty}$ and we claim that

$$w_{\tilde t} \equiv 0 \quad \text{in the connected component of } \Sigma_{\tilde t}^{g_\infty} \text{ containing the point } (0', x_N^\infty). \tag{1.4.16}$$

Indeed, denote by $\mathcal{O}$ the connected component of $\Sigma_{\tilde t}^{g_\infty}$ containing the point $(0', x_N^\infty)$ and let $B = B((0', x_N^\infty), R)$ be any open ball such that $\overline{B} \subset \mathcal{O}$. Then

$$\begin{cases} -\Delta w_{\tilde t} = f(u_{\infty,\tilde t}) - f(u_\infty) \ge -L_f w_{\tilde t} & \text{in} \quad B, \\ w_{\tilde t} \ge 0 & \text{in} \quad B, \\ w_{\tilde t}(0', x_\infty) = 0, \end{cases}$$

where L_f is the Lipschitz constant of f on the compact set $[0, \max_{\overline{B}} u_{\infty,\tilde t}]$ (here we have also used (1.4.10)).
Since $(0', x_N^\infty) \in B$ and (1.4.12) holds, the strong maximum principle ensures that $w_{\tilde t} \equiv 0$ in B and thus, a standard connectedness argument implies (1.4.16).

Since Ω^∞ is an epigraph, the continuity of u_∞ on $\overline{\Omega^\infty}$ and (1.4.16) imply that u_∞ must vanish at some point $\overline{x} \in \Omega^\infty$. This contradicts $u_\infty > 0$ on Ω^∞ and so, the proof of Proposition 1.4.1 is complete.

To conclude the proof of step 2, we use once again our comparison principles. To this end, let us pick $\delta > 0$ such that $3\delta < \min(\frac{\tilde t}{2}, \overline{\varepsilon})$, where $\overline{\varepsilon} = \varepsilon(L_f, \gamma)$, when $f \in Lip([0, +\infty))$ (resp. $\overline{\varepsilon} = \varepsilon(f, M)$, with $M = \sup_{\overline{\Sigma^g_{4\tilde t}}} u$, when $f \in Lip_{loc}([0, +\infty))$).
Since $\delta < \frac{\tilde t}{2}$, by Proposition 1.4.1, there exists $\varepsilon(\delta) \in (0, \delta)$ such that, for any $\varepsilon \in (0, \varepsilon(\delta))$ we have $u \le u_{\tilde t + \varepsilon}$ in $\overline{\Sigma^g_{\delta, \tilde t - \delta}}$.
On $\Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}$, we have $u, u_{\tilde t + \varepsilon} \in H^1_{\text{loc}}(\overline{\Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}}) \cap UC(\overline{\Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}})$ and

$$\begin{cases} -\Delta u - f(u) = 0 = -\Delta u_{\tilde t + \varepsilon} - f(u_{\tilde t + \varepsilon}) & \text{on} \quad \Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}, \\ u \le u_{\tilde t + \varepsilon} & \text{on} \quad \partial\Big(\Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}\Big). \end{cases}$$

For the latter, notice that

$$\partial\Big(\Sigma^g_{\tilde t + \varepsilon} \setminus \overline{\Sigma^g_{\delta, \tilde t - \delta}}\Big) = (\{x_N = \tilde t + \varepsilon\} \cap \overline{\Omega}) \bigcup (\partial\Omega \cap \{x_N < \tilde t + \varepsilon\}) \bigcup \partial(\overline{\Sigma^g_{\delta, \tilde t - \delta}}),$$

and so

- if $x \in \{x_N = \tilde t + \varepsilon\} \cap \overline{\Omega}$ then $u(x) = u_{\tilde t + \varepsilon}(x)$,
- if $x \in \partial\Omega \cap \{x_N < \tilde t + \varepsilon\}$ then $0 = u(x) < u_{\tilde t + \varepsilon}(x)$,

- if $\partial(\overline{\Sigma^g_{\delta,\tilde{t}-\delta}})$, then Proposition 1.4.1 implies that $u(x) \le u_{\tilde{t}+\varepsilon}(x)$.

Notice that $\mathtt{s}_{e_N}(\Sigma^g_{\tilde{t}+\varepsilon}\backslash\overline{\Sigma^g_{\delta,\tilde{t}-\delta}}) \le 2\delta + \varepsilon < 3\delta < \varepsilon(L_f,\gamma)$ and that $\Sigma^g_{\tilde{t}+\varepsilon}\backslash\overline{\Sigma^g_{\delta,\tilde{t}-\delta}} \subseteq \{0 < x_N < \tilde{t}+\varepsilon\}$ (see Figure 1.7). Hence, as in Step 1, we can apply item (i) of Theorem 1.2.1 when $f \in Lip_{loc}([0,+\infty))$ (resp. Theorem 1.2.2 when $f \in Lip([0,+\infty))$) to get

$$u \le u_{\tilde{t}+\varepsilon} \quad \text{in} \quad \Sigma^g_{\tilde{t}+\varepsilon}\backslash\overline{\Sigma^g_{\delta,\tilde{t}-\delta}}.$$

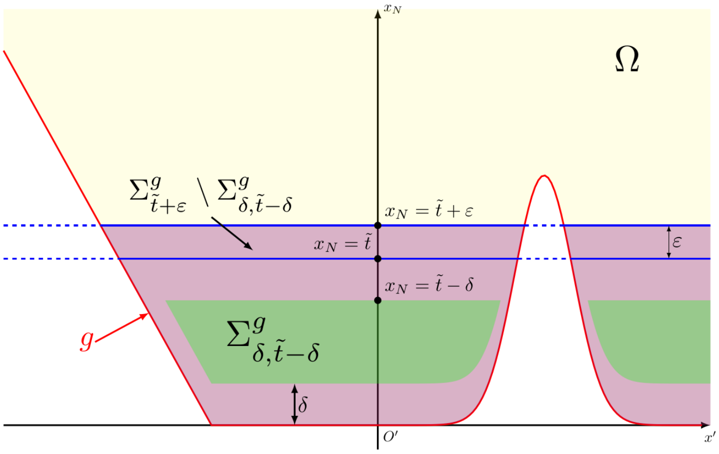

Figure 1.7: Proof of Step 2

The latter and Proposition 1.4.1 imply that $u \le u_{\tilde{t}+\varepsilon}$ in $\Sigma^g_{\tilde{t}+\varepsilon}$. This contradicts the definition of $\tilde{t}$, thus $\tilde{t} = +\infty$.

Step 3 : End of the proof.

For each $t > 0$, let $(x',t) \in \Omega \cap \{x_N = t\}$ and pick $r > 0$ such that $\overline{B((x',t),r)} \subset \Omega$. From the previous step we infer that

$$\begin{cases} -\Delta(u_t - u) + L_{f,t}(u_t - u) \ge 0 & \text{in} \quad \Sigma^g_t \cap B((x',t),r), \\ u_t - u \ge 0 & \text{in} \quad \Sigma^g_t \cap B((x',t),r), \\ u_t - u = 0 & \text{on} \quad \{x_N = t\} \cap B((x',t),r), \end{cases}$$

where $L_{f,t}$ is the Lipschitz constant of f in the compact set $\left[0, \max\limits_{\Sigma^g_t \cap B((x',t),r)} u_t\right]$.

So, by the Hopf lemma, we have

$$-2\tfrac{\partial u}{\partial x_N}(x',t) = \tfrac{\partial(u_t-u)}{\partial x_N}(x',t) < 0. \tag{1.4.17}$$

The latter proves the desired conclusion. □

Proof of Theorem 1.1.3. We follow the proof of Theorem 1.1.1. However, this strategy requires significant changes in Step 2 (recall that Step 2 in the proof of Theorem 1.1.1 was crucially based on the assumption (1.1.5), as well as on the uniform continuity of u on $\Omega \cap \{x_N < R\}$ for any $R > 0$).

Step 1 : Λ is not empty.

It is enough to observe that Step 1 in the proof of Theorem 1.1.1 holds true irrespectively of the value of $f(0)$ and that we can apply Theorem 1.2.2 (see also Remark 1.2.2), since u has at most exponential growth on finite strips by assumption.

Step 2 : $\tilde{t} = \sup \Lambda = +\infty$.

To prove the claim, it is enough to show that Proposition 1.4.1 is still true under the (more general) assumptions of Theorem 1.1.3. Unfortunately, the method used in the proof of Theorem 1.1.1 no longer works if the assumption (1.1.5) is not in force. Indeed, in this case we cannot exclude that the limit profile u_∞ coincides with the function identically equal to 0, and this prevents us from proceeding as before. Also, the fact that u is no longer uniformly continuous on finite strips adds new difficulties. To circumvent those problems we use a different strategy based on translation and scaling arguments.

Proof of Proposition 1.4.1 under the assumptions of Theorem 1.1.3. We proceed as in the proof of Proposition 1.4.1 (in Theorem 1.1.1) until formula (2.4.4) and, for any $k \geq 1$, we consider again $\Omega^k = \{(x', x_N) \in \mathbb{R}^N,\, x_N > g_k(x')\}$ and $\Omega^\infty = \{(x', x_N) \in \mathbb{R}^N,\, x_N > g_\infty(x')\}$.

Since $g_\infty(0') < \tilde{t} - 2\delta$, by the continuity of g_∞ and (2.4.4), we can choose $R > 0$ small enough such that

$$\sup_{x' \in \overline{B'(0', 4R)}} g_\infty(x') < \tilde{t} - \delta, \tag{1.4.18}$$

where $B'(0', 4R) \subset \mathbb{R}^{N-1}$. Since $g_k \to g_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^{N-1})$, there exists $k_1 = k_1(R) \geq 1$ such that

$$\forall k \geq k_1, \quad \forall x' \in \overline{B'(0', 4R)}, \qquad g_\infty(x') - \frac{\delta}{4} \leq g_k(x') \leq g_\infty(x') + \frac{\delta}{4}. \tag{1.4.19}$$

Moreover, in view of (1.4.3),(2.4.4) and (2.4.10) there exists $k_2 = k_2(R) \geq k_1$ such that for any $k \geq k_2$

$$(0', x_N^k) \in \left\{ x = (x', x_N) \in \mathbb{R}^N \,:\, x' \in \overline{B'(0', 2R)}, \quad g_\infty(x') + \frac{\delta}{2} \leq x_N \right\}. \tag{1.4.20}$$

Pick $T > 2\tilde{t} + 2R$ and define the compact set

$$\mathcal{C} = \left(\overline{B'(0', 2R)} \times \mathbb{R} \right) \cap \overline{\Sigma_{2T}^{g_\infty + \frac{\delta}{2}}},$$

and the bounded open set

$$\mathcal{O} = (B'(0', 4R) \times \mathbb{R}) \cap \left\{ x = (x', x_N) \in \mathbb{R}^N \,:\, g_\infty(x') + \frac{\delta}{4} < x_N < 2T + 1 \right\},$$

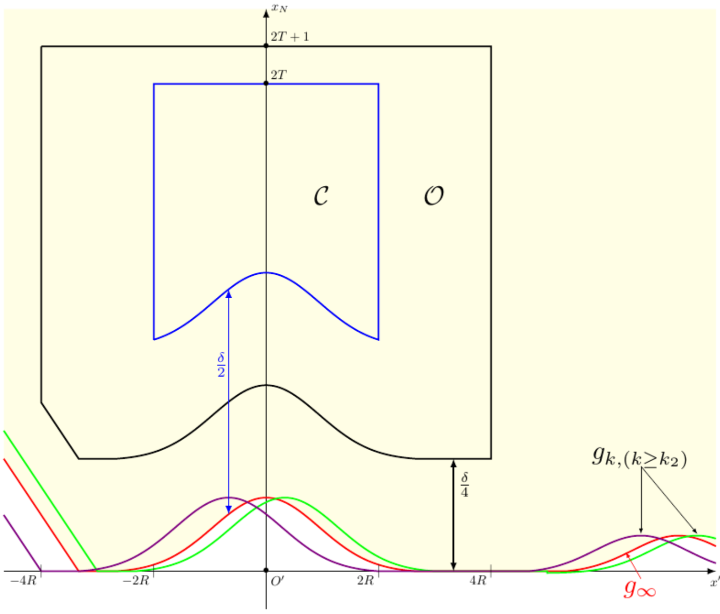

Figure 1.8: $\mathcal{C} \subset\subset \mathcal{O} \subset \Omega^k$

then, by (2.4.10) and (1.4.20), we have (see Figure 1.8)

$$\forall k \geq k_2, \qquad (0', x_N^k) \in \mathcal{C} \subset\subset \mathcal{O} \subset \Omega^k. \tag{1.4.21}$$

Now, for any $k \geq k_2$ and any $x \in \Omega^k$ we define

$$v_k(x) = \frac{u((x^k)' + x', x_N)}{u(x^k)} = \frac{u((x^k)' + x', x_N)}{\alpha_k}, \tag{1.4.22}$$

where $\alpha_k = u(x^k) > 0$, and

$$w_k(x) = v_k(x) + \frac{f(0)}{2\alpha_k}\left(x_N - x_N^k\right)^2. \tag{1.4.23}$$

Then, for any $x \in \mathcal{O}$ we have

$$-\Delta w_k(x) = -\Delta v_k(x) - \frac{f(0)}{\alpha_k} = \frac{f(u((x^k)' + x', x_N)) - f(0)}{\alpha_k} \tag{1.4.24}$$

$$= \frac{f(u((x^k)' + x', x_N)) - f(0)}{\alpha_k w_k(x)} w_k(x) := c_k(x) w_k(x), \tag{1.4.25}$$

where $c_k(x) = \dfrac{f(u((x^k)' + x', x_N)) - f(0)}{\alpha_k w_k(x)}$ is a bounded continuous function satisfying

$$\|c_k\|_{L^\infty(\mathcal{O})} \leq L_f, \qquad \forall\, k \geq k_2. \tag{1.4.26}$$

Indeed, for any $k \geq k_2$ and any $x \in \mathcal{O}$,

$$|c_k(x)| \leq \frac{|f(u((x^k)' + x', x_N)) - f(0)|}{\alpha_k w_k(x)} \leq L_f \frac{u((x^k)' + x', x_N)}{\alpha_k w_k(x)} = L_f \frac{v_k(x)}{w_k(x)} \leq L_f, \tag{1.4.27}$$

since f is globally Lipschitz-continuous and $v_k \le w_k$ in Ω^k, in view of $f(0) \ge 0$.

We also note that

$$w_k(0', x_N^k) = v_k(0', x_N^k) = \frac{u(x^k)}{u(x^k)} = 1, \qquad \forall\, k \ge k_2. \tag{1.4.28}$$

Therefore, for any $k \ge k_2$, the function w_k satisfies

$$\begin{cases} -\Delta w_k = c_k w_k & \text{in} \quad \mathcal{O}, \\ w_k > 0 & \text{in} \quad \mathcal{O}, \\ |c_k| \le L_f & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.4.29}$$

and so we can apply Harnack inequality on the compact set $\mathcal{C}$ to obtain

$$\forall k \ge k_2, \qquad \sup_{\mathcal{C}} w_k \le C_7 \inf_{\mathcal{C}} w_k, \tag{1.4.30}$$

for some constant $C_7 = C_7(N, L_f, \mathcal{C}, \mathcal{O}) > 0$.

By combining (1.4.30), (1.4.28) and (1.4.21) we get that

$$\forall k \ge k_2, \qquad \sup_{\mathcal{C}} v_k \le \sup_{\mathcal{C}} w_k \le C_7 \inf_{\mathcal{C}} w_k \le C_7 w_k(0', x_N^k) \le C_7. \tag{1.4.31}$$

From the latter we also deduce that

$$\forall k \ge k_2, \qquad 0 \le \frac{f(0)}{\alpha_k} \le \frac{2C_7}{\delta^2}, \tag{1.4.32}$$

indeed, since the point $(0', x_N^k + \delta) \in \mathcal{C}$, from (1.4.31) we have

$$0 \le \frac{f(0)\delta^2}{2\alpha_k} \le w_k((0', x_N^k + \delta)) \le C_7. \tag{1.4.33}$$

Furthermore, for $k \ge 1$, if $x \in \Sigma_{\tilde{t}}^{g_k} \cap (\overline{B'(0', 2R)} \times \mathbb{R})$ then $(x' + (x^k)', x_N) \in \Sigma_{\tilde{t}}^{g}$, therefore by definition of Λ we get

$$\frac{\partial u}{\partial x_N}(x' + (x^k)', x_N) > 0,$$

and so, from (1.4.22) we deduce that

$$\forall k \ge 1, \quad \forall x \in \Sigma_{\tilde{t}}^{g_k} \cap (\overline{B'(0', 2R)} \times \mathbb{R}), \qquad \frac{\partial v_k}{\partial x_N}(x) = \frac{1}{\alpha_k}\frac{\partial u}{\partial x_N}(x' + (x^k)', x_N) > 0. \tag{1.4.34}$$

The latter combined with (1.4.31) immediately leads to the following uniform bound

$$\forall k \ge k_2, \quad \forall x \in \overline{\Sigma_{2T}^{g_k} \cap (B'(0', 2R) \times \mathbb{R})}, \qquad v_k(x) \le C_7. \tag{1.4.35}$$

To proceed further we observe that $\Sigma_{2T}^{g_k} \cap (B'(0', 2R) \times \mathbb{R}) = \mathfrak{C}^{g_k}(0', 2R, 2T)$ and that, for any $k \ge k_2$,

$$\begin{cases} -\Delta v_k = \dfrac{f(\alpha_k v_k)}{\alpha_k} = f_k(v_k) & \text{in} \quad \mathfrak{C}^{g_k}(0', 2R, 2T), \\ v_k = 0 & \text{on} \quad \partial\Omega^k \cap (B'(0', R/2) \times \mathbb{R}), \end{cases} \tag{1.4.36}$$

where, for any $t \geq 0$, we have set $f_k(t) = \dfrac{f(\alpha_k t)}{\alpha_k}$. Notice that $f_k \in Lip([0, +\infty))$ with

$$L_{f_k} \leq L_f, \qquad 0 \leq f_k(0) = \frac{f(0)}{\alpha_k} \leq \frac{2C_7}{\delta^2}, \qquad \forall k \geq k_2. \tag{1.4.37}$$

By definition of g_k, all the epigraphs Ω^k satisfy a uniform exterior cone condition on $\partial\Omega^k$ with the same reference cone V (the reference cone for the epigraph Ω), therefore we can apply Proposition 1.3.1 with $h = g_k, \omega = \Omega^k, u = v_k, \widetilde{R} = 2R, \widetilde{\tau} = 2T$, $r = \frac{\widetilde{R}}{4} = \frac{R}{2}, t = \frac{\widetilde{\tau}}{2} = T$ and $k \geq k_2$, to get

$$\exists\, \alpha = \alpha(N, V) \in (0,1) \quad : \quad \forall k \geq k_2 \qquad v_k \in C^{0,\alpha}(\overline{\mathfrak{C}^{g_k}(0', R/2, T)}) \tag{1.4.38}$$

and

$$\|v_k\|_{C^{0,\alpha}(\overline{\mathfrak{C}^{g_k}(0',R/2,T)})} \leq C_5(R,T,V,N)\,(L_{f_k} + |f_k(0)| + 1)\left(\|v_k\|_{L^\infty(\mathfrak{C}^{g_k}(0',2R,2T))} + 1\right) \tag{1.4.39}$$

$$\leq C_5(R,T,V,N)\left(L_f + \frac{2C_7}{\delta^2} + 1\right)(C_7 + 1) = C_8, \tag{1.4.40}$$

for some constant $C_8 = C_8(R, T, V, N, L_f, \delta) > 0$.

Now we consider the compact set $\mathcal{K} = \overline{B'(0', R/2) \times (-T, T)}$ (see Figure 1.9) and, for any $k \geq k_2$, we set

$$\widetilde{v_k}(x) = \begin{cases} v_k(x) & \text{if} \quad x \in \overline{\mathfrak{C}^{g_k}(0', R/2, T)}, \\ 0 & \text{if} \quad x \in \mathcal{K} \setminus \overline{\mathfrak{C}^{g_k}(0', R/2, T)}, \end{cases}$$

then, by (1.4.38) and (1.4.39), $\widetilde{v_k} \in C^{0,\alpha}(\mathcal{K})$ and

$$\|\widetilde{v_k}\|_{C^{0,\alpha}(\mathcal{K})} \leq C_8, \tag{1.4.41}$$

and so, by Ascoli-Arzelà theorem,

$$\exists\, v_\infty \in C^{0,\alpha}(\mathcal{K}) \quad : \quad \widetilde{v_k} \to v_\infty \quad \text{in} \quad C^0(\mathcal{K}), \tag{1.4.42}$$

along a subsequence.

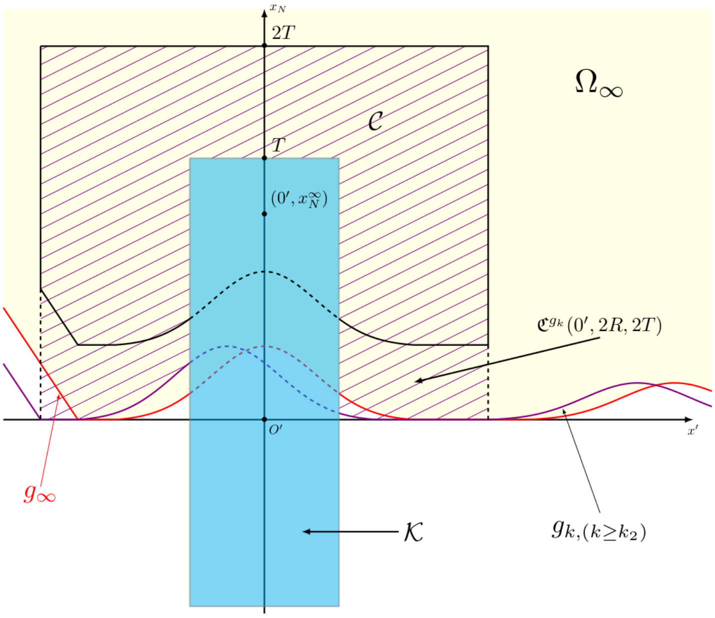

Figure 1.9: $\mathcal{K} = \overline{B'(0', R/2) \times (-T, T)}$

In view of (1.4.36), for any $k \geq k_2$, we have that

$$\begin{cases} -\Delta \widetilde{v_k} = f_k(\widetilde{v_k}) & \text{in} \quad \mathfrak{C}^{g_k}(0', R/2, T) \subset \mathcal{K}, \\ \widetilde{v_k} > 0 & \text{in} \quad \mathfrak{C}^{g_k}(0', R/2, T), \\ \widetilde{v_k} = 0 & \text{on} \quad \partial\Omega^k \cap (B'(0', R/2) \times \mathbb{R}), \\ \widetilde{v_k}(0', x_N^k) = v_k(0', x_N^k) = 1. \end{cases} \tag{1.4.43}$$

Since $f_k \in Lip([0, +\infty))$ and (1.4.37) is in force, another application of Ascoli-Arzelà theorem tell us that, up to a subsequence,

$$f_k \to f_\infty \quad \text{in} \quad C^0_{\text{loc}}([0, +\infty)), \tag{1.4.44}$$

for some $f_\infty \in Lip([0, +\infty))$ satisfying $f_\infty(0) \geq 0$.

Now we recall that, $x_N^k \longrightarrow x_N^\infty \in \left[g_\infty(0') + \delta, \tilde{t} - \delta\right]$, $g_k \longrightarrow g_\infty$ in $C^0_{\text{loc}}(\mathbb{R}^{N-1})$ and that, (1.4.42) and (1.4.44) are in force, therefore we can pass to the limit, as $k \longrightarrow +\infty$, into (1.4.43) to get

$$\begin{cases} -\Delta v_\infty = f_\infty(v_\infty) & \text{in} \quad \mathcal{D}'(\mathfrak{C}^{g_\infty}(0', R/2, T)), \\ v_\infty \geq 0 & \text{in} \quad \mathfrak{C}^{g_\infty}(0', R/2, T), \\ v_\infty = 0 & \text{on} \quad \partial\Omega^\infty \cap (B'(0', R/2) \times \mathbb{R}), \\ (0', x_N^\infty) \in \mathfrak{C}^{g_\infty}(0', R/2, T), & \quad v_\infty(0', x_N^\infty) = 1. \end{cases} \tag{1.4.45}$$

By standard interior regularity theory for elliptic equations we see that v_∞ is a classical solution to (2.4.24) and the strong maximum principle then implies

$$v_\infty > 0 \qquad \text{in} \qquad \mathfrak{C}^{g_\infty}(0', R/2, T). \tag{1.4.46}$$

By letting $k \longrightarrow +\infty$ into (1.4.34), and recalling the definition of $\widetilde{v_k}$, we deduce that

$$v_\infty(x) \le v_{\infty,\tilde{t}}(x) \qquad \forall x \in \overline{\Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R})}. \tag{1.4.47}$$

We notice that (1.4.2) implies $u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k)$, for any $k \ge 1$. Then, by definition of v_k and $\widetilde{v_k}$, we deduce that $\widetilde{v_k}(0', x_N^k) > \widetilde{v_k}_{,\tilde{t}+\varepsilon_k}(0', x_N^k)$, for any $k \ge 1$. Passing to the limit in the latter we deduce

$$v_\infty(0', x_N^\infty) \ge v_{\infty,\tilde{t}}(0', x_N^\infty),$$

and so

$$v_\infty(0', x_N^\infty) = v_{\infty,\tilde{t}}(0', x_N^\infty)$$

since $(0', x_N^\infty) \in \Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R})$ and (2.4.26) holds true.

Now, for any $x \in \Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R})$ we set

$$w_\infty(x) = v_{\infty,\tilde{t}}(x) - v_\infty(x),$$

then w_∞ satisfy

$$\begin{cases} -\Delta w_\infty = f_\infty(v_{\infty,\tilde{t}}) - f_\infty(v_\infty) \ge -L_f w_\infty & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R}), \\ w_\infty \ge 0 & \text{in} \quad \Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R}), \\ (0', x_N^\infty) \in \Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R}), & \quad w_\infty(0', x_N^\infty) = 0. \end{cases} \tag{1.4.48}$$

By applying the strong maximum principle to (2.4.27), we infer that $w_\infty \equiv 0$ on the domain $\Sigma_{\tilde{t}}^{g_\infty} \cap (B'(0', R/2) \times \mathbb{R})$. Therefore, we have

$$0 = v_\infty(x', g_\infty(x')) = v_\infty(x', 2\tilde{t} - g_\infty(x')), \quad \forall\, x' \in B'(0', R/2),$$

which contradicts (1.4.46), since $0 < 2\tilde{t} - g_\infty(x') < T$ whenever $x' \in B'(0', R/2)$. This completes the proof of Proposition 1.4.1 when $f(0) = 0$.

The remaining part of the proof of Step 2 is the same of the one of Step 2 in Theorem 1.1.1, just observe that one can use Theorem 1.2.2 (see also Remark 1.2.2), since u has at most exponential growth on finite strips by assumption.

Step 3 is unchanged. □

Proof of Theorem 1.1.2. The proof of Theorem 1.1.2 is a straightforward adaptation of the one of Theorem 1.1.3. Indeed, any step of the proof of Theorem 1.1.3 remains unchanged if one observes that

i) one can apply Theorem 1.2.1, since by assumption u is bounded on any finite strip and f is a locally-Lipschitz continuous function,

ii) Step 2 only requires that u is bounded on any finite strip and that f belongs to $Lip_{loc}([0, +\infty))$ and satisfies $f(0) \ge 0$. Indeed, those properties imply the validity of (1.4.26) and (1.4.27), where L_f is replaced by the Lipschitz constant of f on the interval $[0, M]$. Here, we have set $M := \sup_{\Omega \cap \{x_N < 2T+1\}} u$, which is finite by assumption.

Step 3 is unchanged, since it only requires that f belongs to $Lip_{loc}([0, +\infty))$. □

Proof of Corollary 1.1.1. Since ∇u is bounded on Ω, $u \in C^0(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, it is straightforward to check that u satisfies $|u(x) - u(y)| \leq \|\nabla u\|_\infty |x - y|$ for any $x, y \in \Omega$. Then, u is uniformly continuous on Ω and $u \in W^{1,\infty}_{loc}(\overline{\Omega}) \subseteq H^1_{loc}(\overline{\Omega})$. Furthermore, the boundedness of ∇u implies (via the mean value theorem) that u is bounded on finite strips. The conclusion then follows by applying Theorem 1.1.1. $\square$

Proof of Corollary 1.1.2. $u \in W^{1,\infty}_{loc}(\overline{\Omega}) \subseteq H^1_{loc}(\overline{\Omega})$, since $u \in C^0(\overline{\Omega})$, $u = 0$ on $\partial\Omega$ and ∇u is bounded in Ω by assumption. The latter assumption also implies (via the mean value theorem) that u is bounded on finite strips. The conclusion then follows by applying Theorem 1.1.2. $\square$

Proof of Corollary 1.1.3. Since $\partial\Omega$ is locally of class $C^{1,\alpha}$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, standard regularity results for elliptic equations implies that $u \in C^{1,\alpha}_{loc}(\overline{\Omega})$. Therefore $u \in H^1_{loc}(\overline{\Omega})$ and the conclusion then follows from Theorem 1.1.2. $\square$

Proof of Theorem 1.1.4. The proof is similar to the one of Theorem 1.1.1, but it is easier, since the coercivity assumption on g implies that any cap Σ^g_θ is a bounded open set of $\mathbb{R}^N$. Thus, we provide only the modifications necessary to deal with the new points of the proof related to the fact that g is supposed to be merely continuous and that no restriction on the sign of $f(0)$ is imposed.
Since every cap Σ^g_θ is bounded, the comparison principle immediately gives the conclusion of Step 1.

To achieve the conclusion of Step 2 we need to show that Proposition 1.4.1 still holds under the assumptions of Theorem 1.1.4.
To this end we first observe that the sequence of points (x^k) appearing in (1.4.2) is bounded (since the cap $\Sigma^g_{\tilde{t}-\delta}$ is bounded) and so, up to a subsequence, we may and do suppose that $x^k \to x^\infty := ((x^\infty)', x^\infty_N)$, as $k \to \infty$. Therefore, the sequence of continuous functions (g_k) defined by (1.4.4) is again relatively compact in $C^0_{loc}(\mathbb{R}^{N-1})$. Hence, up to a subsequence, $g_k \to g_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^{N-1})$. Note that $g_\infty(x') = g(x' + (x^\infty)')$ for any $x' \in \mathbb{R}^{N-1}$.

Moreover, the sequence $(\tilde{u}_k)$ defined by (2.4.6) is again relatively compact in $C^0_{loc}(\mathbb{R}^N)$. This is because the sequence $(\tilde{u}_k(0', -1))$ is bounded in $\mathbb{R}$ and the function $\tilde{u}$ is bounded and uniformly continuous on any finite strip. Hence, up to a subsequence, $\tilde{u}_k \to \tilde{u}_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^N)$, where $\tilde{u}_\infty(x) = \tilde{u}(x' + (x^\infty)', x_N)$ for any $x \in \mathbb{R}^N$.

Thanks to those information, we can follow the proof until (1.4.12). Now, if we set $w_{\tilde{t}} := u_{\infty,\tilde{t}} - u_\infty$ on $\Sigma^{g_\infty}_{\tilde{t}}$, we see that

$$\begin{cases} -\Delta w_{\tilde{t}} = f(u_{\infty,\tilde{t}}) - f(u_\infty) \geq -L_f w_{\tilde{t}} & \text{in } \ \Sigma^{g_\infty}_{\tilde{t}}, \\ \qquad\quad w_{\tilde{t}} \geq 0 & \text{in } \ \Sigma^{g_\infty}_{\tilde{t}}, \\ \qquad w_{\tilde{t}}(0', x_\infty) = 0, & \end{cases} \tag{1.4.49}$$

where now L_f denotes the Lipschitz-constant of f on the closed interval $\left[0, \max_{\overline{\Sigma^{g_\infty}_{\tilde{t}}}} u_{\infty,\tilde{t}}\right]$ (note that $\overline{\Sigma^{g_\infty}_{\tilde{t}}}$ is a compact set, since g_∞ is coercive). Since $(0', x^\infty_N) \in \Sigma^{g_\infty}_{\tilde{t}}$, the strong maximum principle and the continuity of $w_{\tilde{t}}$ ensure that

$$w_{\tilde{t}} \equiv 0 \quad \text{in the connected component of } \Sigma^{g_\infty}_{\tilde{t}} \text{ containing the point } (0', x^\infty_N).$$

Since Ω^∞ is an epigraph, the latter implies that u_∞ must vanish at some point of Ω^∞. But (in view of the form of g_∞ and $\tilde{u}_\infty$) the latter implies that the solution u must vanish at some point of Ω. A contradiction. This proves the claim of Proposition 1.4.1.

The remaining part of the proof of Step 2 is the same of the one of Step 2 in Theorem 1.1.1, since u is bounded on finite strips.

Step 3 is unchanged, since it only requires that f belongs to $Lip_{loc}([0, +\infty))$. □

1.5 Extensions to merely continuous epigraphs bounded from below and further observations

In this section we prove some monotonicity results for solutions of (1.1.1) on certain merely continuous epigraphs Ω bounded from below, i.e.,where the function g defining $\partial\Omega$ is continuous but not necessarily uniformly continuous. To this end we observe that the uniform continuity of the function g enters into the proofs of Theorems 1.1.1-1.1.2 only to use the following classical compactness result (via the Ascoli-Arzelà theorem): *let (g_k) be a sequence of translations of a uniformly continuous $g : \mathbb{R}^{N-1} \to \mathbb{R}$, i.e., $g_k(\cdot) = g(\cdot + x_k)$, for some sequence (x_k) of points of $\mathbb{R}^{N-1}$. If the sequence (g_k) is bounded at a fixed point of $\mathbb{R}^{N-1}$, then it admits a subsequence converging uniformly on every compact sets of $\mathbb{R}^{N-1}$*
.
Therefore, all we need for the above mentioned proofs to work, is that the continuous function $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is bounded from below and it satisfies the following compactness property :

$(\mathcal{P})$ *Any sequence (g_k) of translations of g, which is bounded at some fixed point of $\mathbb{R}^{N-1}$, admits a subsequence converging uniformly on every compact sets of $\mathbb{R}^{N-1}$.*

The above discussion leads to the following

Definition 1.5.1. Assume $N \geq 2$. We say that a continuous function $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ belongs to the class $\mathcal{G}$, if it satisfies the compactness property $(\mathcal{P})$.

Before stating the new monotonicity results, let us show how large the class $\mathcal{G}$ is. Hereafter, we provide a wide (but non-exhaustive) list of members of $\mathcal{G}$.

1. Uniformly continuous functions on $\mathbb{R}^{N-1}$ belong to $\mathcal{G}$.

2. Coercive continuous functions on $\mathbb{R}^{N-1}$ belong to $\mathcal{G}$.[12]

   In particular, any continuous function on $\mathbb{R}^{N-1}$ such that $\lim_{|x|\mapsto\infty} g(x) \in (-\infty, +\infty]$ belongs to $\mathcal{G}$.

3. Let us denote by $\mathsf{G}(\mathbb{R}^{N-1})$ the set of continuous functions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ enjoying the following property : *there exists a* continuous bijection *$\phi : \mathbb{R} \to \mathbb{R}$ such that*

[12]Since the sequence $(g_k) := (g(\cdot + x_k))$ is bounded at some point of $\mathbb{R}^{N-1}$ and g is coercive, it follows that the sequence (x_k) is bounded in $\mathbb{R}^{N-1}$. Therefore (g_k) is uniformly equicontinuous on compact sets of $\mathbb{R}^{N-1}$, and the compactness follows from the Ascoli-Arzelà theorem.

$\phi \circ g \in \mathcal{G}$.
It is immediate to check that $\mathsf{G}(\mathbb{R}^{N-1}) \subset \mathcal{G}$.[13] Moreover, the family $\mathsf{G}(\mathbb{R}^{N-1})$ strictly contains the one of *uniformly continuous functions* on $\mathbb{R}^{N-1}$ and the one of *coercive continuous functions* on $\mathbb{R}^{N-1}$, as shown by the next examples.[14]

(3a) For instance, the functions $g_1(x_1) = e^{x_1}$ if $N = 2$, and $g(x) = e^{x_1 + \sum_{j=2}^{N-1} \cos^j(x_j)}$ if $N \geq 3$, are neither uniformly continuous nor coercive on $\mathbb{R}^{N-1}$. Nevertheless, they belong to the class $\mathsf{G}(\mathbb{R}^{N-1})$ (consider, for instance, the function

$$\phi(t) = \begin{cases} t & \text{if} \quad t \leq 0, \\ \log(t+1) & \text{if} \quad t > 0, \end{cases}$$

in the previous definition and observe that $\phi \circ g \in \mathcal{G}$ is uniformly continuous, hence $\phi \circ g \in \mathcal{G}$). The same argument also proves that $g(x_1) = e^{e^{x_1}}$ and $g(x) = e^{e^{x_1 + \sum_{j=2}^{N-1} \cos^j(x_j)}}$ do belong to $\mathsf{G}(\mathbb{R}^{N-1})$. Since this argument can be iterated, we see that the class $\mathsf{G}(\mathbb{R}^{N-1})$ contains smooth functions, bounded from below, that are neither uniformly continuous nor coercive, and with arbitrary large growth at infinity. Also note that, g_1 being convex, its epigraph satisfies a uniform exterior cone condition.

(3b) Assume $N \geq 2$ and let $g \in C^2(\mathbb{R}^{N-1})$ be any positive function such that $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$, then $\sqrt{g}$ is globally Lipschitz-continuous on $\mathbb{R}^{N-1}$ (see Lemma *I* in [Gla63] for $N = 2$). Therefore, we have $g \in \mathsf{G}(\mathbb{R}^{N-1})$.[15] More generally, any $g \in C^2(\mathbb{R}^{N-1})$, bounded from below and such that $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$ is a member of the class $\mathsf{G}(\mathbb{R}^{N-1})$.
For instance, the function $g = g(x_1, \dots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1} \sin(j x_j)$ belongs to $\mathsf{G}(\mathbb{R}^{N-1})$ for any $N \geq 3$.

4. For $N = 2$, any continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\ell_- := \lim_{x \mapsto -\infty} g(x) \in (-\infty, +\infty]$ and $\ell_+ := \lim_{x \mapsto +\infty} g(x) \in (-\infty, +\infty]$ belongs to $\mathcal{G}$. Therefore, any quasiconvex (resp. quasiconcave) continuous function bounded from below belongs to $\mathcal{G}$. In particular, any monotone continuous function bounded from below and any convex functions bounded from below belongs to $\mathcal{G}$.

5. Assume $2 \leq n < N$ and let $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be a member of $\mathcal{G}$. Then, the function $\tilde{g} : \mathbb{R}^{N-1} \to \mathbb{R}$ defined by $\tilde{g}(x_1, \dots, x_{N-1}) = g(x_1, \dots, x_{n-1})$ satisfies the compactness property $(\mathcal{P})$, as a function on $\mathbb{R}^{N-1}$. Therefore, $\tilde{g} \in \mathcal{G}$, as a function on $\mathbb{R}^{N-1}$.

[13] Assume that (g_k) is bounded at some point of $\mathbb{R}^{N-1}$, say $\bar{x}$, then the sequence of functions $((\phi \circ g)_k) := ((\phi \circ g)(\cdot + x_k))$ is bounded at $\bar{x}$. Therefore, up to a subsequence, $(\phi \circ g)_k \to \varphi$ uniformly on compact sets of $\mathbb{R}^{N-1}$, since $\phi \circ g \in \mathcal{G}$ by assumption. But then we also have $g_k = \phi^{-1}((\phi \circ g)_k) \to \phi^{-1} \circ \varphi$ uniformly on compact sets of $\mathbb{R}^{N-1}$.

[14] Taking the identity function as ϕ, we immediately see that uniformly continuous functions, as well as coercive continuous functions $g : \mathbb{R}^{N-1} \to \mathbb{R}$ do belong to $\mathsf{G}(\mathbb{R}^{N-1})$.

[15] Choose the function

$$\phi(t) = \begin{cases} -\sqrt{-t} & \text{if} \quad t < 0, \\ \sqrt{t} & \text{if} \quad t \geq 0, \end{cases}$$

In particular, for $N \geq 2$, the functions $g(x_1, \dots, x_N) = e^{e^{x_1}}$ and $g(x_1, \dots, x_N) = e^{x_1} - 4\arctan(x_1) - 2$ belong to $\mathcal{G}$, are bounded from below and their epigraphs satisfy a uniform exterior cone condition.

6. Assume $N \geq 2$. Let $g \in \mathcal{G}$ and let $T : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ be a transformation of the form $T(x) = Ax + b$, where A an invertible real matrix and $b \in \mathbb{R}^{N-1}$. Then, $g \circ T \in \mathcal{G}$. In particular, this results applies when T is an isometry of $\mathbb{R}^{N-1}$.

7. Assume $N \geq 2$ and $\lambda \geq 0$. Let $g, \tilde{g} \in \mathcal{G}$ be bounded from below. Then, $\lambda g \in \mathcal{G}$ and $g + \tilde{g} \in \mathcal{G}$.
   In particular, for $N \geq 2$, the function $g(x_1, \dots, x_N) = x_1^4 + e^{x_2}$ belongs to $\mathcal{G}$, is bounded from below and its epigraph satisfies a uniform exterior cone condition.

8. By combining items (1)-(7), one can easily build further examples of functions g belonging to $\mathcal{G}$ in any dimension $N \geq 2$. In particular, one can construct members of $\mathcal{G}$ bounded from below, that are neither uniformly continuous nor coercive, with arbitrary large growth at infinity, and such that their epigraphs satisfy a uniform exterior cone condition.

In view of the above discussions, we can now state the monotonicity results for continuous epigraphs defined by functions g belonging to the class $\mathcal{G}$. Let us start with the extension of Theorem 1.1.1 and Corollary 1.1.1.

Theorem 1.5.1. *Let $N \geq 2$ and let Ω be an epigraph bounded from below and defined by a function $g \in \mathcal{G}$. Assume $f \in Lip_{loc}([0, +\infty))$ with*

$$\liminf_{t \to 0^+} \frac{f(t)}{t} > 0. \tag{1.5.1}$$

(i) If $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ is a distributional solution to (1.1.1) *which is bounded and uniformly continuous on finite strips.*
Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.
If f is globally Lipschitz-continuous, the same conclusion holds true only under the sole assumption that u is uniformly continuous on finite strips.

(ii) If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical solution of (1.1.1) *such that $\nabla u \in L^\infty(\Omega)$. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

If we also assume that the epigraph satisfies a uniform exterior cone condition, then we can prove the following extensions of Theorem 1.1.3 and Theorem 1.1.2.

Theorem 1.5.2. *Let $N \geq 2$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition.*

(i) Assume $f \in Lip([0, +\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) *with at most exponential growth on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

(ii) Assume $f \in Lip_{loc}([0,+\infty))$ with $f(0) \geq 0$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to (1.1.1) *which is bounded on finite strips. Then u is strictly increasing in the x_N-direction, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.*

Note that Theorem 1.5.1 and Theorem 1.5.2 apply to the explicite examples of functions g provided in the above items (1)-(7). They also recover the results in the very recent preprint [GMS24], where the monotonicity is proved for some classical solutions to (1.1.1) with $f(t) = t^q$ and $q \geq 1$.[16]

The next result applies to solutions of (1.1.1) where Ω is merely a continuous epigraph and f is a non-increasing function, possibly discontinuous, and with no restriction on the sign of $f(0)$.

Theorem 1.5.3. *Let Ω be any continuous epigraph bounded from below and let $f : [0,\infty) \mapsto \mathbb{R}$ be any non-increasing function. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (1.1.1) *with subexponential growth on finite strips* [17]*.*
Then u is non-decreasing, i.e., $\frac{\partial u}{\partial x_N} \geq 0$ in Ω.[18]
Moreover, if $f \in Lip_{loc}([0,+\infty))$, then u is strictly increasing, i.e., $\frac{\partial u}{\partial x_N} > 0$ in Ω.

Proof. The proof is a straightforward application of Theorem 1.2.3. Indeed, for any $\theta > 0$ we have $u, u_\theta \in H^1_{\mathrm{loc}}(\overline{\Sigma^g_\theta}) \cap C^0(\overline{\Sigma^g_\theta})$ and

$$\begin{cases} -\Delta u - f(u) = 0 = -\Delta u_\theta - f(u_\theta) & \text{in } \Sigma^g_\theta, \\ u \leq u_\theta & \text{on } \partial\Sigma^g_\theta. \end{cases}$$

Since u and u_θ have subexponential growth on Σ^g_θ (which is bounded in the direction e_N) we can apply Theorem 1.2.3 to get that

$$u \leq u_\theta \quad \text{in} \quad \Sigma^g_\theta.$$

Therefore, u is non-decreasing in the x_N-direction.
If f is locally Lipschitz-continuous, we conclude as in Step 3 of Theorem 1.1.2. □

Some remarks are in order.

Remark 1.5.1. (i) We note that the proof of Theorem 1.5.3 also applies to some unbounded domains that are not necessarily epigraphs. Indeed, given any domain $\Omega \subset \mathbb{R}^N$ bounded from below, it is immediate to see that the proof of Theorem 1.5.3 applies if Ω contains

[16] It is immediate to see that any semicoercive and continuous function is bounded from below and belongs to our class $\mathcal{G}$ (see items (5) and (6) above). This observation and Lemma A.2 in [GMS24] also show that the convex epigraph case is covered by the techniques we have developed in the present article.

[17] i.e., for any $R > 0$,

$$\limsup_{\substack{|x|\to\infty, \\ x \in \Omega \cap \{x_N < R\}}} \frac{\ln u(x)}{|x|} \leq 0.$$

[18] Note that $u \in C^1(\Omega)$, since $f(u) \in L^\infty_{loc}(\Omega)$.

the reflection (with respect to the hyperplane $\{x_N = \theta\}$) of any cap $\Omega \cap \{x_N < \theta\}$. Since the latter property is clearly satisfied if $\Omega \subsetneq \mathbb{R}^N$ supports a monotone solution to (1.1.1), we see that the above result actually caracterizes the euclidean proper domains for which the homogeneous Dirichlet BVP (1.1.1) admits a monotone solution.[19]
For example, the domain $\Omega_4 = \bigcup_{k\in\mathbb{Z}}\{x \in \mathbb{R}^2 : |x_1 - (3+4k)| < 1\,,\ x_2 > -1\} \bigcup \{x \in \mathbb{R}^2 : x_2 > 0\}$ (see Figure 1.10) and the open orthant $\Omega_5 = \{x \in \mathbb{R}^N : x_1 > 0, \dots, x_N > 0\}$ satisfy this property, but they are not epigraphs with respect to e_N (in fact, Ω_4 is never an epigraph, that is, there is no unit vector ν of $\mathbb{R}^2$ that allows it to be represented as an epigraph with respect to ν).
Another example is depicted in Figure 1.11.

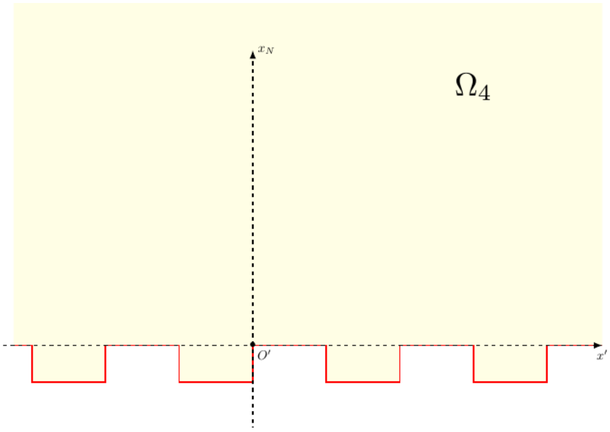

Figure 1.10: Ω_4

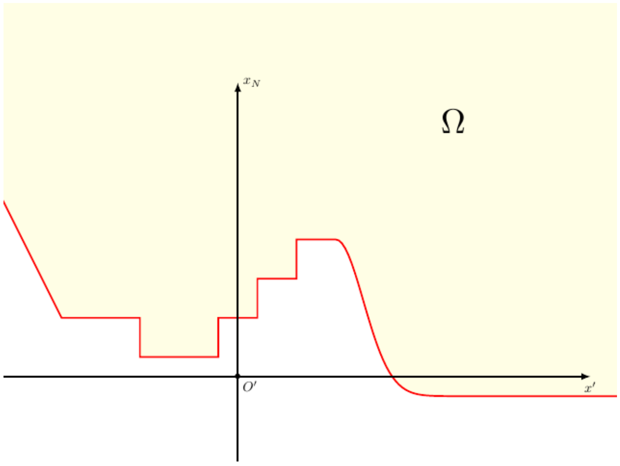

Figure 1.11:

(ii) We also observe that the first conclusion of Theorem 1.5.3, namely that u is non-decreasing, is sharp. See the explicit example 1.7.2 in Section 1.7.

We conclude this section with the following general result, which we believe to be of independent interest. This is a straightforward application of our comparison principles (see Step 1 in the proof of our monotonicity theorems). It shows that the moving plane method can be started irrespectively of the value of $f(0)$ and for a large class of unbounded domains. Specifically we have the following

Corollary 1.5.1 (Starting the moving planes method). *Assume $N \geq 2$. Let Ω be a domain contained in the upper half-space $\mathbb{R}^N_+$ and also assume that Ω contains the reflection (with respect to the hyperplane $\{x_N = R\}$) of a finite strip $\Omega \cap \{x_N < R\} \not\equiv \emptyset$. Let u be a distributional (resp. classical) solution to* (1.1.1). *Suppose that one of the following assumptions is in force :*

[19] this can be formulated in the following equivalent way : Problem (1.1.1) with f non-increasing, admits a monotone solution on a domain $\Omega \subsetneq \mathbb{R}^N$, if and only if, Ω is contained in an affine half-space (whose inner normal is denoted by ν) and, for any $x \in \Omega$ the open half-line $\{x + t\nu,\, t > 0\}$ is contained in Ω. That is, if and only if, Ω is contained in an affine half-space (whose inner normal is denoted by ν) and Ω is ν-invariant.

(i) $f \in Lip_{loc}([0,+\infty))$ *and* u *is bounded on* $\Omega \cap \{x_N < R\}$*;*

(ii) $f \in Lip([0,+\infty))$ (*resp. a non-increasing function*) *and* u *has at most exponential growth on* $\Omega \cap \{x_N < R\}$*.*

Then there exists $R_0 \in (0, R)$ *such that* $\Omega \cap \{x_N < R_0\} \not\equiv \emptyset$ *and*

$$\frac{\partial u}{\partial x_N} > 0 \quad in \quad \Omega \cap \{x_N < R_0\}\,.$$

The above result extends the one proved in [Far20] for the special case of a half-space.

For sake of clarity (and simplicity) we have stated the above result in the case $\Omega \subseteq \mathbb{R}^N_+$. Since the considered problem is invariant by isometry, it is clear that the same result holds if Ω is contained in an affine open half-space H. In this case, the monotonicity will be obtained with respect to ν, the inner normal to H.

1.6 Some applications to classification and non-existence results

In this section, we apply our monotonicity results to prove some classification and nonexistence results for the problem (1.1.1) on general continuous epigraphs defined by a function $g \in \mathcal{G}$ (see Section 1.5).

Theorem 1.6.1. *Let* Ω *be an epigraph defined by a function* $g \in \mathcal{G}$*. Also suppose that* Ω *is bounded from below and satisfies a uniform exterior cone condition.*
Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ *be a bounded distributional solution to*

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega, \\ u \geq 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{1.6.1}$$

Assume that $f \in C^1([0,+\infty))$*,* $f(t) > 0$ *for* $t > 0$ *and* $2 \leq N \leq 11$*, then* $u \equiv 0$ *and* $f(0) = 0$*.*

Proof. By standard interior regularity theory for elliptic equations u belongs to $C^2(\Omega)$ hence, either $u \equiv 0$ in Ω or $u > 0$ in Ω by the strong maximum principle. Let us rule out the second case. Suppose $u > 0$ in Ω, then we can apply Theorem 1.5.2 to prove that $\dfrac{\partial u}{\partial x_N} > 0$ in Ω. Hence, u is a bounded stable solution to (1.6.1) of class $C^2(\Omega)$ and so, the function

$$v(x') = \lim_{x_N \to +\infty} u(x', x_N), \qquad\qquad x' \in \mathbb{R}^{N-1},$$

is a positive stable classical solution to $-\Delta v = f(v)$ in $\mathbb{R}^{N-1}$, with $N - 1 \leq 10$. We can therefore apply Theorem 1 in [DF22] to get that $v \equiv const. = a > 0$ et $f(a) = 0$. Since the latter contradicts the positivity assumption on f, we infer that $u \equiv 0$ and so $f(0) = 0$. $\square$

The previous theorem remains true even for $N \geq 12$, if we add an assumption about the behaviour of f at the origin. In this case, the desired conclusion is obtained by making use of some Liouville-type theorems for stable solutions established in [Far07], [DF10] and [Far15] (instead of Theorem 1 in [DF22]).
The desired results are the contents of Theorem 1.6.2 and Theorem 1.6.3 below.

Theorem 1.6.2. *Assume $N \geq 12$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to* (1.6.1) *where $f \in C^1([0,+\infty))$ satisfies $f(t) > 0$ for $t > 0$ and $\liminf_{t\to 0^+} \frac{f(t)}{t^s} > 0$, for some $s \in \left[0, \frac{N-3}{N-5}\right)$. Then $u \equiv 0$ and $f(0) = 0$.*

Proof. By proceeding as in the proof of Theorem 1.6.1, if $u > 0$ in Ω we can construct $v \in C^2(\mathbb{R}^{N-1})$ which is a positive, bounded, stable solution to $-\Delta v = f(v)$ in $\mathbb{R}^{N-1}$. Now, pick $z > \sup_{\mathbb{R}^{N-1}} v > 0$ and consider the new nonlinear function $\tilde{f} \in C^1([0,+\infty))$ defined as follows : $\tilde{f} = f$ on $[0, \sup_{\mathbb{R}^{N-1}} v]$, $\tilde{f} > 0$ on $[\sup_{\mathbb{R}^{N-1}} v, z)$, $\tilde{f} \leq 0$ on $[z, +\infty)$.[20] Hence, v is also a bounded stable solution to

$$\begin{cases} -\Delta v = \tilde{f}(v) & \text{in} \quad \mathbb{R}^{N-1}, \\ \quad v > 0 & \text{on} \quad \mathbb{R}^{N-1}. \end{cases} \tag{1.6.2}$$

Since $\liminf_{t\to 0^+} \frac{\tilde{f}(t)}{t^s} = \liminf_{t\to 0^+} \frac{f(t)}{t^s} > 0$ and $\tilde{f} > 0$ on $(0, \sup_{\mathbb{R}^{N-1}} v)$, we see that

$$\exists\, \delta_1 > 0 \quad : \quad \forall\, x \in \mathbb{R}^{N-1} \qquad \tilde{f}(v(x)) \geq \delta_1 v^s(x).$$

Therefore we can apply Theorem 1.2 of [Far15] to v (with $M = N-1$) and find that $v \equiv 0$. A contradiction. Hence $u \equiv 0$ and $f(0) = 0$. $\square$

Notice that Theorem 1.6.1 and Theorem 1.6.2 immediately imply the following non-existence result when $f(0) > 0$.

Corollary 1.6.1. *Assume $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be a uniformly continuous epigraph bounded from below and satisfying a uniform exterior cone condition.
If $f \in C^1([0,+\infty))$ satisfies $f(t) > 0$ for $t \geq 0$, then problem* (1.6.1) *does not admit any bounded distributional solution of class $C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$.*

The next result concerns the following natural class of nonlinearities introduced in [DF10]:

$$\begin{cases} f \in C^1([0,+\infty)) \cap C^2((0,+\infty)), \quad f(0) = 0, \\ \\ f > 0, \ \textit{nondecreasing and convex in } (0,+\infty) \\ \\ \qquad \textit{s.t.} \qquad \lim_{u\to 0^+} \frac{f'(u)^2}{f(u)f''(u)} := q_0 \in [0,+\infty] \end{cases} \tag{1.6.3}$$

where, in the latter, we agree to set $\frac{f'(u)^2}{f(u)f''(u)} = +\infty$ if $f''(u) = 0$.
Notice that, we necessarily have $q_0 \in [1,+\infty]$ (see Lemma 1.4 in [DF10]).

[20] Note that, thanks to the assumptions on f and z, such a function $\tilde{f}$ does exist.

The typical representative of this class is given by the function $f(u) = u^p$, $p > 1$. In this case $\frac{f'(u)^2}{f(u)f''(u)} = \frac{p}{p-1}$ and so q_0 coincides with the conjugate exponent of p. Consequently, if we define $p_0 \in [1, +\infty]$ as the conjugate exponent of q_0 by $\frac{1}{p_0} + \frac{1}{q_0} = 1$, we have that the exponent p_0 can be considered as a "measure" of the flatness of f at the origin.
Other members of the preceding class are provided by the functions $f_n(u) = e^u - \sum_{k=0}^{n} \frac{u^k}{k!}$, where $n \geq 1$ is an integer. It is easily seen that $q_0(f_n) = \frac{n+1}{n}$ and so $p_0(f_n) = n + 1$.

Theorem 1.6.3. *Assume $N \geq 12$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition. Let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a bounded distributional solution to* (1.6.1) *where f satisfies* (1.6.3)*.*
Suppose that p_0, the conjugate exponent of q_0, satisfies

$$1 \leq p_0 < p_c(N-1), \tag{1.6.4}$$

where p_c is the Joseph-Lundgren stability exponent given by

$$p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

Then $u \equiv 0$.

Note that the preceding theorem applies to $f(u) = u^p$ if $1 \leq p < p_c(N-1)$ and to f_n if $n + 1 < p_c(N-1)$.

Proof of Theorem 1.6.3. By proceeding as in the proof of Theorem 1.6.1, if $u > 0$ in Ω we can construct $v \in C^2(\mathbb{R}^{N-1})$ which is a positive, bounded, stable solution to $-\Delta v = f(v)$ in $\mathbb{R}^{N-1}$. Since the assumption (1.6.4) is in force, we can apply Theorem 1.5 in [DF10] to get that $v \equiv$ *const.* $= a > 0$ et $f(a) = 0$. A contradiction, hence $u \equiv 0$. $\square$

Next we state the following immediate consequence of Theorem 1.1.4.

Corollary 1.6.2. *When the epigraph Ω is coercive, the conclusion of Theorems 1.6.1-1.6.3 and Corollary 1.6.1 holds true under the sole assumption of continuity of g, i.e., we do not need to require the uniform exterior cone condition for the epigraph.*

Theorems 1.6.1-1.6.3 and Corollary 1.6.2 recover and extend/complement some results in [Far07], [DF22], [CLZ14], [DFP23] and [GMS24].

We conclude this section with the following classification result for solutions of (1.6.1) that tend to zero at infinity.

Theorem 1.6.4. *Assume $N \geq 2$ and let Ω be an epigraph defined by a function $g \in \mathcal{G}$. Also suppose that Ω is bounded from below and satisfies a uniform exterior cone condition. Assume $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a distributional solution to* (1.6.1) *such that*

$$\lim_{\substack{x \in \Omega, \\ |x| \longrightarrow \infty}} u(x) = 0. \tag{1.6.5}$$

Then $u \equiv 0$ and $f(0) = 0$.
When the epigraph Ω is coercive, the above conclusion holds true under the sole assumption of continuity of g.

Note that, in the previous result, we make no assumptions on f (beside $f \in Lip_{loc}([0, +\infty))$) or the boundedness of u.

The above result recovers and improves upon a result of [EL82], where the conclusion has been obtained for a smooth coercive epigraph.

Proof of Theorem 1.6.4. First we prove that u is bounded and then we use this information to deduce that $f(0) = 0$. The boundedness of u easily follows from the continuity of u on $\overline{\Omega}$ and (1.6.5). Then, pick an open ball $B \subset\subset \Omega$ and, for any integer $k \geq 1$ and any $x \in B$, consider the function $u_k(x) = u(x', x_N + k)$. By the boundedness of u and standard elliptic estimates, we have that a subsequence of (u_k) (still denoted by (u_k)) converges in the $C^2_{loc}(B)$-topology to a solution $v \in C^2(B)$ of $-\Delta v = f(v)$ in B. Since (1.6.5) is in force, we necessarily have $v \equiv 0$ which, in turn, yields $f(0) = 0$. By the latter and the strong maximum principle, either $u \equiv 0$ in Ω or $u > 0$ in Ω. Let us prove that the second case cannot occur. Since u is bounded and $f(0) = 0$, we can apply Theorem 1.5.2 (resp. Theorem 1.1.4, when Ω is coercive) to prove that $\dfrac{\partial u}{\partial x_N} > 0$ in Ω. But the latter contradicts the assumption (1.6.5). Hence $u \equiv 0$ and $f(0) = 0$. $\square$

1.7 Some examples

Example 1.7.1. *Let us consider the following two functions g_1, $g_2 : \mathbb{R} \mapsto \mathbb{R}$ defined by*

$$g_1(x) = \begin{cases} 0 & \text{if } x \in (-\infty, -4], \\ \sqrt{4 - (x+2)^2} & \text{if } x \in [-4, 0], \\ \sqrt{4 - (x-2)^2} & \text{if } x \in [0, 2], \\ 2 & \text{if } x \in [2, +\infty), \end{cases}$$

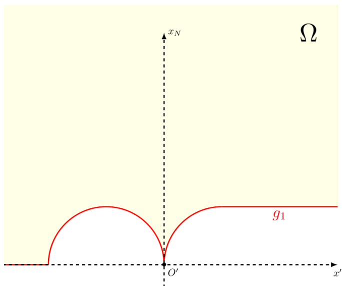

Figure 1.12: g_1

$$g_2(x) = \begin{cases} 0 & \text{if } x \in (-\infty, -4], \\ \sqrt{4 - (x+2)^2} & \text{if } x \in [-4, 0], \\ \sqrt{4 - (x-2)^2} & \text{if } x \in [0, 2], \\ 2 & \text{if } x \in [2, 6], \\ x - 4 & \text{if } x \in [6, +\infty). \end{cases}$$

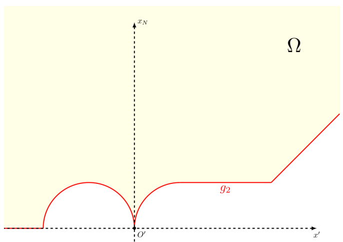

Figure 1.13: g_2

Note that g_1 is globally $\frac{1}{2}$-Hölder-continuous and g_2 is locally $\frac{1}{2}$-Hölder-continuous, but neither g_1 nor g_2 are locally α-Hölder-continuous, for any $\alpha \in (\frac{1}{2}, 1]$. Also observe that $g_2(x) = g_1(x) + (x-6)^+$ for any $x \in \mathbb{R}$, therefore the two-dimensional epigraphs defined by g_1 and g_2 are uniformly continuous, bounded from below, but not locally Lipschitz-continuous. Moreover, it is easily seen that both of them satisfy a uniform exterior sphere condition (of radius $\frac{1}{2}$). Hence, they also satisfy a uniform exterior cone condition.

Example 1.7.2. *Assume $N \geq 2$ and let Ω be the half-space $\{x \in \mathbb{R}^N : x_N > 0\}$. The bounded function*

$$u(x) = \begin{cases} 1 - (x_N - 1)^4 & \text{if } 0 \leq x_N \leq 1, \\ 1 & \text{if } x_N > 1, \end{cases} \tag{1.7.1}$$

is a classical C^2 solution to (1.1.1), *where f is given by the following non-increasing function*

$$f(t) = \begin{cases} 12 & \text{if } t < 0, \\ 12\sqrt{1-t} & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1. \end{cases} \tag{1.7.2}$$

Note that f is globally Hölder-continuous, but not locally Lipschitz-continuous on $\mathbb{R}$.

The following example (inspired from [Far01]) shows that we cannot remove the assumption on the Lipschitz character of f in our main results.

Example 1.7.3. *Assume $N \geq 2$ and let Ω be the half-space $\{x \in \mathbb{R}^N : x_N > 0\}$. The bounded function (see Figure 1.14)*

$$u(x) = \begin{cases} (1 - (x_N - 1)^4)^4 & \text{if } 0 \leq x_N \leq 1, \\ 1 & \text{if } 1 < x_N \leq 2, \\ 1 + (1 - (x_N - 3)^4)^4 & \text{if } 2 < x_N \leq 4, \\ 1 & \text{if } x_N > 4. \end{cases} \tag{1.7.3}$$

is a classical C^2 solution to (1.1.1), *where f is given by the following globally Hölder-continuous function*

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ -192(t(1-t^{\frac{1}{4}}))^{\frac{1}{2}}(1 - \frac{5}{4}t^{\frac{1}{4}}) & \text{if } 0 \leq t \leq 1, \\ -192((t-1)(1-(t-1)^{\frac{1}{4}}))^{\frac{1}{2}}(1 - \frac{5}{4}(t-1)^{\frac{1}{4}}) & \text{if } 1 < t \leq 2, \\ 0 & \text{if } t > 2. \end{cases} \tag{1.7.4}$$

Note that f is not locally Lipschitz-continuous on $\mathbb{R}$ and that $\frac{\partial u}{\partial x_N}$ changes sign on $\mathbb{R}^N_+$.

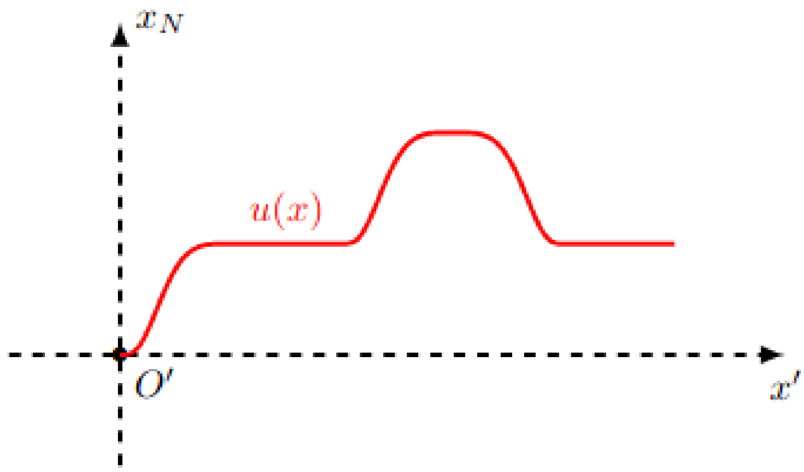

Figure 1.14: Graph of u

Chapter 2

Serrin's overdetermined problems on epigraphs

Contents

2.1 Introduction

In his seminal paper [Ser71], published in 1971, J. Serrin proved that if Ω is a bounded and smooth domain for which there is a smooth solution to the overdetermined problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial \eta} = \mathfrak{c} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.1}$$

then Ω must be a ball and u is radially symmetric about its center.
Here, η denotes the outward unit normal at $\partial\Omega$ and f is a Lipschitz-continuous function.

In this paper we are concerned with Serrin's overdetermined problem (2.1.1) on epigraphs, i.e., on unbounded domains of the form

$$\Omega := \{x = (x', x_N) \in \mathbb{R}^N, x_N > g(x')\}, \tag{2.1.2}$$

where $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a (sufficiently) smooth function and $N \geq 2$.

Our study is motivated by the following question : *if f is a globally Lipschitz-continuous function and the overdetermined problem* (2.1.1) *admits a smooth and bounded solution on a smooth epigraph Ω, then Ω must be an affine half-space.* This is a special case of a question raised in 1997 by H. Berestycki, L. Caffarelli and L. Nirenberg at the end of their work [BCN97b]. In the following, we shall refer to this problem as the *conjecture BCNe*.

In [BCN97b], for functions f of Allen-Cahn type[1], the authors employed the sliding method to prove the following rigidity result : *if $g \in C^2$ is a globally Lipschitz-continuous function satisfying an additional condition at infinity,*[2] *and problem* (2.1.1) *admits a smooth and bounded solution, then g must be constant* (*i.e.,* $\Omega = \{x \in \mathbb{R}^N : x_N > const.\}$ *is an upper half-space*) *and u takes the form $u = u(x_N)$.*

This gives a partial answer to the above conjecture. Indeed, since problem (2.1.1) is invariant by euclidean isometries, all the natural solutions of (2.1.1) corresponding to the affine functions $g(x) = a \cdot x + b$, $a \neq 0$, are ruled out by the additional condition on g at infinity.

In 2010, A. Farina and E. Valdinoci [FV10] introduced a new geometric tool that allowed them to prove the *conjecture BCNe* without any additional condition on g, but under the dimensional constraint $N \leq 3$ (see Theorem 1.8 in [FV10]). Specifically, they proved the following : *if f is of Allen-Cahn type, $g \in C^3$ is a globally Lipschitz-continuous function and problem* (2.1.1) *admits a bounded C^2-solution, then Ω is an affine half-space and u is one-dimensional.*

Moreover, in [FV10] it is proved that :

i) *the same result is true if Ω is a two-dimensional domain of class C^3 (not necessarily an epigraph), f is any locally Lipschitz-continuous function and u is a monotone solution* (*i.e.* $\frac{\partial u}{\partial x_2} > 0$ *on* Ω) *with bounded gradient;*

[1] i.e., $f \in Lip_{loc}([0, +\infty))$ such that :

$$\begin{cases} & \exists\, \mu > \delta_2 > \delta_1 > 0 \ : \\ \text{(I)} & f > 0 \quad \text{on } (0, \mu), \qquad f \leq 0 \quad \text{on } [\mu, +\infty), \\ \text{(II)} & f(t) \geq \delta_1 t \quad \text{for } t \in (0, \delta_1), \qquad f \text{ is non-increasing on } (\delta_2, \mu). \end{cases} \tag{2.1.3}$$

The typical representative is given by the classical Allen-Cahn nonlinearity $f(t) = t - t^3$.

[2] The additional condition on g required in [BCN97b] is : $\forall \tau \in \mathbb{R}^{N-1}$, $\lim_{|x'| \to +\infty} (g(x'+\tau) - g(x')) = 0$.

ii) *for any locally Lipschitz-continuous function f, Serrin's overdetermined problem has no bounded solution on globally Lipschitz-continuous coercive epigraphs if either $N = 2$ or $N = 3$ and $f \geq 0$.*

Subsequently, the new geometric tool introduced in [FV10] has been used to study Serrin's overdetermined problem on general domains of a Riemannian manifold (see [FMV13]).

In [WW19], under more specific conditions on f (but still including the classical Allen-Cahn equation), K. Wang and J. Wei proved that the *conjecture BCNe* is true :

iii) *on any smooth epigraph, if $N = 2$;*

iv) *on any smooth and globally Lipschitz-continuous epigraph, if $N \geq 3$;*

v) *on any smooth epigraph, if $N \leq 8$ and u is a monotone solution, i.e. $\frac{\partial u}{\partial x_N} > 0$ on Ω.*

vi) *Serrin's overdetermined problem has no solution on smooth coercive epigraphs if $N \geq 2$.*

The result in item v) above is optimal since M. Del Pino, F. Pacard and J. Wei have built a smooth epigraph Ω (which is not an affine half-space) such that, if $N \geq 9$, the overdetermined problem (2.1.1) admits a smooth, bounded and monotone solution (see Theorem 1 in [PPW15]). Also note that, the epigraph built in [PPW15] is not globally Lipschitz-continuous.

The proofs in [WW19] involve tools from Geometric Measure Theory and the theory of free boundary problems.

Finally we recall that the authors of [RRS17] proved the *conjecture BCNe* for *any two-dimensional epigraph of class $C^{1,\alpha}$ and any Lipschitz-continuous function f, if $\mathfrak{c} < 0$ in* (2.1.1).

In this paper, we prove the validity of the *conjecture BCNe* in dimension $N \geq 2$ in several cases. Our results extend and complement the known results mentioned above. In broad terms, we prove the *conjecture BCNe* when the epigraph is bounded from below and, for possibly unbounded solutions. Our results are new, even for bounded solutions.

When $f(0) \geq 0$, they are crucially based on the *monotonicity* results of our recent paper [BFS25] in which we prove that, when Ω is an epigraph bounded from below, any solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.4}$$

is *monotone*, i.e., u satisfies $\frac{\partial u}{\partial x_N} > 0$ on Ω.

The situation when $f(0) < 0$ is more involved. The only known results in this case are restricted to dimension 2. In the present work, to deal with the case $f(0) < 0$, we first prove a new monotonicity result, valid in any dimension and for solutions to (2.1.1) on epigraphs bounded from below, and then deduce the validity of the *conjecture BCNe* for $N \leq 3$.

Specifically, in Section 2.2 we consider Serrin's overdetermined problem (2.1.1) for a family of specified Allen-Cahn-type nonlinearities. In this framework, we prove the validity of the *conjecture BCNe* in any dimension $N \geq 2$, for any solution to (2.1.1) and for a large class of epigraphs bounded from below (larger than that of globally Lipschitz-continuous epigraphs). This is the content of Theorem 2.2.3. In passing, we also establish a general rigidity result for monotone solutions to (2.1.1) (see Theorem 2.2.1). This result interpolates between the above mentioned results iii)-vi). It is of independent interest and ancillary to Theorem 2.2.3

In Section 2.3 we consider a general locally Lipschitz-continuous f satisfying $f(0) \geq 0$. In this framework we prove the *conjecture BCNe* in dimension $N \leq 3$, for possibly unbounded solutions u (see Theorems 2.3.2, 2.3.3 and 2.3.4). The necessity to consider also unbounded solutions to the Serrin's overdetermined problem (2.1.1) comes from the fact that, for some natural functions f, the only solutions to (2.1.1) are unbounded. This is well illustrated by the case of harmonic functions, that is when $f \equiv 0$. In this case, there might be no bounded solution to (2.1.1), whereas the function $u(x) = x_N$ is a positive unbounded harmonic function on the half-space $\mathbb{R}^N_+$ with zero Dirichlet boundary condition (see Proposition 2.5.1 and Remark 2.5.1).

The proofs are obtained by combining the geometric approach developed by one of the authors in [FV10] and the monotonicity results we recently obtained in [BFS25].

Section 2.4 deals with the case of locally Lipschitz-continuous functions f with $f(0) < 0$. As already mentioned, we first prove a new monotonicity result in any dimension (cf. Theorem 2.4.2) and then we deduce the validity of the *conjecture BCNe* for $N \leq 3$, using the geometric approach developed by one of the authors in [FV10] (see Theorems 2.4.3 and 2.4.4).

Several results used to prove our main conclusions are established in Section 2.5. Some of these general results are of independent interest and could be useful for further studies of the nonlinear Poisson equation on unbounded domains. For the reader's convenience, the notations used in the paper are collected at the end of Section 2.5.

We end this section by observing that, in the present work, we always consider domains Ω of class C^1 and solutions u belonging to $C^1(\overline{\Omega}) \cap C^2(\Omega)$. These are the minimal assumptions on Ω and on u to deal with the problem in a classical framework. We also note that, when $\mathfrak{c} \neq 0$ in (2.1.1), these assumptions automatically ensure that Ω is of class $C^{2,\alpha}$ and $u \in C^{2,\alpha}_{loc}(\overline{\Omega})$, for any $\alpha \in (0,1)$ (cf. Theorem 1 of [Vog92] and observe that the proof of this result is local). These results will therefore be used in our work (perhaps without explicitly recalling them).

2.2 Serrin's overdetermined problem with Allen-Cahn type nonlinearity

In this section we investigate the validity of the *conjecture BCNe* when f is modeled on the classical Allen-Cahn nonlinearity. More precisely, we deal with a family of nonlinearities f, already considered in [WW19], which we will call *specified Allen-Cahn-type nonlinearities.*

A function f is a *specified Allen-Cahn-type nonlinearity* if the function

$$W(t) := \int_0^1 f(s)ds - \int_0^t f(s)ds, \qquad t \geq 0, \tag{2.2.1}$$

belongs to $C^2([0, +\infty))$ and satisfies the following conditions :

W1) $W \geq 0$, $W(1) = 0$ and $W > 0$ in $[0, 1)$;

W2) $W' < 0$ in $(0, 1)$, and either $W'(0) \neq 0$ or

$$W'(0) = 0 \quad \text{and} \quad W''(0) \neq 0;$$

W3) $\exists\, \delta_2 \in (0, 1), \exists\, \kappa > 0$ such that $W'' \geq \kappa$ in $[\delta_2, 1]$;

W4) $\exists\, p > 1, \exists\, c > 0$ such that $W'(t) \geq c(t-1)^p$ for $t > 1$.

The representative example of this family is given by $f(t) = t - t^3$, the well-known *Allen-Cahn nonlinearity.*[3]

For this family of nonlinear functions, the authors of [WW19] have proved results iii), iv), v) and vi) previously discussed in the Introduction. Our first theorem interpolates between those results. It applies to monotone solutions, it is of independent interest and it will be ancillary to the main result of this section.

Theorem 2.2.1. *Assume $N \geq 2$ and let $f \in C^1([0, +\infty))$ be a specified Allen-Cahn type nonlinearity. Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to* (2.1.1) *satisfying*

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega. \tag{2.2.2}$$

Let $\Omega \subset \mathbb{R}^N$ be an epigraph defined by a function $g \in C^1(\mathbb{R}^{N-1})$ and, if $N \geq 9$ let us also assume that g satisfies :

a) $\exists\, C > 0$ *for which*

$$g(x') \geq -C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1};$$

or

b) $\exists\, C > 0$ *for which*

$$g(x') \leq C(1 + |x'|) \qquad \forall\, x' \in \mathbb{R}^{N-1}.$$

[3] in this case we have $W(t) = \frac{(1-t^2)^2}{4}$.

Then, $\Omega = \mathbb{R}^N_+$ *up to isometry and there exists* $u_0 : [0, +\infty) \to [0, +\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+. \tag{2.2.3}$$

Some remarks are in order

Remark 2.2.1. (i) In the above Theorem we do not assume that u is bounded. Indeed, this property is automatically satisfied by any solution to (2.1.4) on general domains, possibly unbounded, when f is a specified Allen-Cahn type nonlinearity. See Corollary 2.5.1.

(ii) When $N \leq 8$, the above Theorem recovers Theorem 1.4 in [WW19] (that is, result v) discussed in the introduction).

We also notice that, when g is globally Lipschitz-continuous, the monotonicity assumption (2.2.2) is automatically satisfied in any dimension $N \geq 2$ thanks to item (i) and the work [BCN97b]. Since the Lipschitz character of g implies the validity of the additional condition a) (actually also b)), we see that the above Theorem also recovers Theorem 1.2 in [WW19] (that is, result iv) discussed in the introduction).

When g is coercive, the monotonicity assumption (2.2.2) is automatically satisfied in any dimension $N \geq 2$ thanks to the work [EL82] and, since the additional condition a) is clearly in force, we see that also Theorem 1.3 in [WW19] (that is, result vi) discussed in the introduction) is a particular case of Theorem 2.2.1 above.

(iii) The additional assumptions for $N \geq 9$ have the following geometrical interpretation: either $\mathbb{R}^N \setminus \Omega$ contains a cone or Ω contains a cone (or equivalenty, either the graph of g lies above the graph of a cone or the graph of g lies below the graph of a cone).

(iv) The above remarks also show that the epigraph constructed by M. Del Pino, F. Pacard and J. Wei in [PPW15] does not contain a cone (and neither does its complementary) and therefore that this epigraph is neither globally Lipschitz-continuous (as already observed in [PPW15]), nor contained in an affine half-space.

In view of the discussions in Remark 2.2.1 above, the following two research lines seem legitimate and interesting, when f is a *specified Allen-Cahn-type nonlinearities* :

1) for $N \leq 8$, provide natural assumptions on g, weaker than the global Lipschitz continuity, which guarantee the monotonicity property;

2) for $N \geq 9$, give natural assumptions on the epigraph Ω in order to restore the validity of *conjecture BCNe*.

The next result takes a step in the directions indicated in items 1) and 2) above. In general terms, this result proves the validity of *conjecture BCNe* for epigraphs contained in a half-space and without the monotonicity assumption. To our knowledge, this is the first result of this type in any dimension $N \geq 3$. It crucially builds on the results of our recent paper [BFS25] in which we prove the monotonicity property (2.2.2) for solutions to (2.1.4) on epigraphs bounded from below, and defined by a function g belonging to a large class $\mathcal{G}$ of continuous functions. Specifically, in [BFS25] we introduced the following

Definition 2.2.2. Assume $N \geq 2$. We say that a continuous function $g : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ belongs to the class $\mathcal{G}$, if it satisfies the next compactness property :

($\mathcal{P}$) *Any sequence (g_k) of translations of g, which is bounded at some fixed point of $\mathbb{R}^{N-1}$, admits a subsequence converging uniformly on every compact sets of $\mathbb{R}^{N-1}$*

and we showed that, in particular, the following continuous functions belong to $\mathcal{G}$ (see Section 5 of [BFS25] for more information, properties and examples about this class) :

1. Uniformly continuous functions on $\mathbb{R}^{N-1}$ (not necessarily globally Lipschitz-continuous) belong to $\mathcal{G}$.

2. For $N \geq 3$. Any function $g \in C^2(\mathbb{R}^{N-1})$, bounded from below and such that $\nabla^2 g \in L^\infty(\mathbb{R}^{N-1})$ is a member of the class $\mathcal{G}$.
   For instance, $g = g(x_1, \ldots, x_{N-1}) = (x_1)^2 + \prod_{j=2}^{N-1} \sin(jx_j)$ belongs to $\mathcal{G}$.

3. For $N = 2$, any continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\ell_- := \lim_{x \mapsto -\infty} g(x) \in (-\infty, +\infty]$ and $\ell_+ := \lim_{x \mapsto +\infty} g(x) \in (-\infty, +\infty]$ belongs to $\mathcal{G}$. Therefore, any quasiconvex (resp. quasiconcave) continuous function bounded from below belongs to $\mathcal{G}$. In particular, *any monotone continuous function bounded from below* belongs to $\mathcal{G}$ as well as *any convex functions bounded from below.*

4. Further explicit examples of member of $\mathcal{G}$ are provided by the following functions (see Section 5 of [BFS25]) : $g(x_1) = e^{x_1}$ if $N = 2$ and $g(x) = e^{x_1 + \sum_{j=2}^{N-1} \cos^j(x_j)}$ if $N \geq 3$. Also, $g(x_1, \ldots, x_N) = e^{e^{x_1}}$ and $g(x_1, \ldots, x_N) = x_1^4 + e^{x_2}$ qualify for any $N \geq 2$.

We are now ready for the main result of this section, which provides an extension of the known results mentioned in the Introduction (following the two research lines described above).

Theorem 2.2.3. *Assume $N \geq 2$ and let Ω be an epigraph bounded from below and defined by a function $g \in \mathcal{G} \cap C^1(\mathbb{R}^{N-1})$.*
Let $f \in C^1([0, +\infty))$ be a specified Allen-Cahn type nonlinearity and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (2.1.1).

Then, $\Omega = \mathbb{R}^N_+$ up to a vertical translation and there exists $u_0 : [0, +\infty) \to [0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \qquad \forall x \in \mathbb{R}^N_+. \tag{2.2.4}$$

Proof. On the one hand, condition `W2`) implies that $\lim_{t \to 0^+} \frac{f(t)}{t} > 0$. On the other hand, u is bounded by Corollary 2.5.1. Hence, ∇u is bounded too, thanks to Corollary 2.5.2. We can therefore apply Theorem 5.1. of [BFS25] to get that $\frac{\partial u}{\partial x_N} > 0$ on Ω. Since Ω satisfies condition a), being bounded from below, the desired conclusion then follows by applying Theorem 2.2.1. $\square$

We conclude this section with the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. To establish the desired conclusion we shall use the results proved by K. Wang and J. Wei in [WW19]. We first observe that, up to a translation of Ω in the vertical direction, we may and do suppose that the origin 0 of $\mathbb{R}^N$ belongs to $\partial\Omega$. Then, following [WW19], we recall that any solution to (2.1.1) satisfying the monotonicity assumption (2.2.2) (when extended by 0 outside Ω) is a (local) minimizer of the functional

$$\int \frac{|\nabla u|^2}{2} + W(u)\mathbf{1}_{\{u>0\}}$$

in $\mathbb{R}^N$, which also satisfies $0 \leq u < 1$. Consequently, for a subsequence $\varepsilon_n \longrightarrow 0^+$, the blowing down sequence $u_n(x) := u(\frac{x}{\varepsilon_n})$ converges in $L^1_{loc}(\mathbb{R}^N)$ to the characteristic function $\mathbf{1}_{\Omega_\infty}$, where Ω_∞ is a non-trivial set of least perimeter in $\mathbb{R}^N$ (i.e., a set of least perimeter with $0 \in \partial\Omega_\infty$). Furthermore, on any connected compact set of $\mathbb{R}^N \setminus \partial\Omega_\infty$, either $u_n \longrightarrow 1$ uniformly or $u_n \equiv 0$ for all large n.

At this point, for any $x \in \mathbb{R}^N$, we set $v(x) = u(x', -x_N)$, therefore $v_n(x) = u_n(x', -x_N)$,

$$\{v > 0\} = \Big\{(x', x_N) \in \mathbb{R}^N \ : \ x_N < -g(x')\Big\},$$

$$\{v_n > 0\} = \Big\{(x', x_N) \in \mathbb{R}^N \ : \ x_N < \varphi_n(x')\Big\} := U_n$$

where

$$\varphi_n(x') = -\varepsilon_n g\left(\frac{x'}{\varepsilon_n}\right) \qquad x' \in \mathbb{R}^{N-1}.$$

Also observe that v_n converges in $L^1_{loc}(\mathbb{R}^N)$ to $\mathbf{1}_{-\Omega_\infty}$ and that on any connected compact set of $\mathbb{R}^N \setminus \partial(-\Omega_\infty)$, either $v_n \longrightarrow 1$ uniformly or $v_n \equiv 0$ for all large n. Hence

$$\mathbf{1}_{U_N} \longrightarrow \mathbf{1}_{-\Omega_\infty} \quad \text{in } L^1_{loc}(\mathbb{R}^N).$$

The latter and Lemma 16.3 in [Giu84] tell us that the set $-\Omega_\infty$ is itself the subgraph of a mesurable function $\varphi_\infty : \mathbb{R}^{N-1} \mapsto [-\infty, +\infty]$, which is the a.e.-limit of φ_n (up to a subsequence).

To obtain the conclusion of the Theorem, it is enough to prove that the blowing down limit Ω_∞ is a half-space (see Theorem 1.6 in [WW19]). To this end, below we perform the classical blow-down analysis for sets of least perimeter (see for instance the book of E. Giusti [Giu84]). Specifically, since Ω_∞ is a non-trivial set of least perimeter in $\mathbb{R}^N$, the blow-down sequence $\frac{\Omega_\infty}{n}$ converges in $L^1_{loc}(\mathbb{R}^N)$ to a non-trivial minimal cone $\mathcal{C}$. We also recall that this procedure tell us that, the limit cone $\mathcal{C}$ is a half-space if and only if Ω_∞ is a half-space (see e.g. Theorem 17.3 and Theorem 9.3 [Giu84]).

Hence, to conclude the proof of the Theorem it is enough to prove that $\mathcal{C}$ is a half-space. To this end, we first claim that

$$\mathcal{C} + e_N \subseteq \mathcal{C}, \tag{2.2.5}$$

where e_N denotes the unit vector $(0, \cdots, 1) \in \mathbb{R}^N$. Indeed, as Ω_∞ is a subgraph, it follows that $\Omega_\infty + te_N \subseteq \Omega_\infty$ for every $t > 0$. This yields that $\frac{\Omega_\infty}{n} + e_N \subseteq \frac{\Omega_\infty}{n}$ for every n. Hence, by $L^1_{loc}(\mathbb{R}^N)$ convergence, $\mathcal{C} + e_N \subseteq \mathcal{C}$. By (2.2.5) and Lemma 2.3 of [GMM97] (cf. also Proposition 2.1 of [Far18]) we see that either $\mathcal{C}$ is a half-space or it is a cylinder of the form $\mathcal{C} = \mathcal{C}' \times \mathbb{R}$, for some non-trivial minimal cone $\mathcal{C}' \subseteq \mathbb{R}^{N-1}$.

When $N - 1 \leq 7$, all the non-trivial minimal cones are half-spaces (see [Giu84]), hence for $N \leq 8$ we have that $\mathcal{C}$ is a half-space and so the Theorem is proved.

Now we turn to the case $N \geq 9$ and we first suppose that g satisfies the assumption $a)$. Under this assumption we see that $\mathbb{R}^N \setminus \mathcal{C}$ contains the cone

$$\mathfrak{C} := \left\{(x', x_N) \in \mathbb{R}^N \,:\, x_N > -C|x'|\right\}.$$

Indeed, for almost every $x' \in \mathbb{R}^{N-1}$,

$$\varphi_\infty(x') \longleftarrow \varphi_n(x') \leq \varepsilon_n C \left(1 + \left|\frac{x'}{\varepsilon_n}\right|\right) \longrightarrow C|x'|,$$

that is

$$\varphi_\infty(x') \leq C|x'| \quad \text{for almost every } x' \in \mathbb{R}^{N-1}, \tag{2.2.6}$$

which mmediately implies that $\mathfrak{C} \subseteq \mathbb{R}^N \setminus \Omega_\infty$.

Also, since $-\Omega_\infty$ is the subgraph of φ_∞, we see that $\mathbf{1}_{-\frac{\Omega_\infty}{n}} \longrightarrow \mathbf{1}_{-C}$ in $L^1_{loc}(\mathbb{R}^N)$. Therefore, $-C$ is itself the subgraph of a mesurable function $\psi_\infty : \mathbb{R}^{N-1} \mapsto [-\infty, +\infty]$, which is the a.e.-limit of $(\varphi_\infty)_n$ (up to a subsequence) defined by

$$(\varphi_\infty)_n(x') = \varepsilon_n \varphi_\infty \left(\frac{x'}{\varepsilon_n}\right) \qquad x' \in \mathbb{R}^{N-1}.$$

The latter and (2.2.6) clearly imply that $\mathfrak{C} \subseteq \mathbb{R}^N \setminus \mathcal{C}$.

If $\mathcal{C}$ is not a half-space, then $\mathcal{C} = \mathcal{C}' \times \mathbb{R}$, for some non-trivial minimal cone $\mathcal{C}' \subseteq \mathbb{R}^{N-1}$. The latter and the inclusion $\mathfrak{C} \subseteq \mathbb{R}^N \setminus \mathcal{C}$ imply that $\mathcal{C}' = \mathbb{R}^{N-1}$, contradicting its non-triviality. This concludes the proof when g satisfies the assumption a). A similar argument provides the desired result under condition b). $\square$

2.3 Serrin's overdetermined problem with general nonlinearity. The case $f(0) \geq 0$.

In this section we consider a general locally Lipschitz-continuous f with $f(0) \geq 0$. In this framework we prove the *conjecture BCNe* in dimension $N \leq 3$, for possibly unbounded solutions u.

The proofs are obtained by combining the geometric approach developed by one of the authors in [FV10] and the monotonicity results we recently obtained in [BFS25].

Let us start with a general rigidity result for monotone solutions to the overdetermined problem (2.1.1), when Ω is a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem 2.3.1. *Assume $N = 2, 3$ and let $\Omega \subset \mathbb{R}^N$ be a domain of class C^3. Let $f \in Lip_{loc}([0, +\infty))$ and let $u \in C^2(\overline{\Omega})$ be a solution to (2.1.1) such that*

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad \text{as } R \longrightarrow \infty. \tag{2.3.1}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{2.3.2}$$

then, $\Omega = \mathbb{R}^N_+$ up to isometry and there exists $u_0 : [0, +\infty) \to (0, +\infty)$ strictly increasing such that

$$u(x) = u_0(x_N) \quad \forall x \in \mathbb{R}^N_+.$$

Remark 2.3.1. Note explicitly that in the above Theorem :

(i) no restriction is imposed on the sign of $f(0)$ or on $\mathfrak{c}$;

(ii) Ω is not assumed to be an epigraph.

Proof. We use the geometric approach developped in [FV10]. To this end we first recall that, given a smooth function v, one may consider the level sets $\{v = c\}$. In the vicinity of $\{\nabla v \neq 0\}$, these level sets are smooth manifolds, so one can introduce the principal curvatures

$$k_1, \cdots, k_{N-1}$$

at any point of such manifolds. We set

$$\mathcal{K} := \sqrt{k_1^2 + \cdots + k_{N-1}^2}.$$

Also, we consider the tangential gradient along level sets of v at these points, that is

$$\nabla_T v := \nabla v - \Big(\nabla v \cdot \frac{\nabla v}{|\nabla v|}\Big)\frac{\nabla v}{|\nabla v|}.$$

With these notations, and given that u is monotone with respect to the vertical direction, we can apply Theorem 1.1 of [FV10] to obtain

$$\int_\Omega (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2)\phi^2 \leq \int_\Omega |\nabla u|^2 |\nabla \phi|^2, \tag{2.3.3}$$

for any compactly supported, Lipschitz-continuous function $\phi : \mathbb{R}^N \mapsto \mathbb{R}$.

Now, for any $R \geq 1$ and $x \in \mathbb{R}^N$ we define

$$\phi_R(x) := \mathbf{1}_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R}\mathbf{1}_{B_R \setminus B_{\sqrt{R}}}(x).$$

Then, ϕ_R can be inserted into (2.3.3) to get

$$\int_{\Omega\cap B_{\sqrt{R}}} |\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2 \leq \frac{4}{\ln^2 R}\int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega} \frac{|\nabla u(x)|^2}{|x|^2}dx. \tag{2.3.4}$$

Moreover, since $\int_{|x|}^{R}\frac{2}{t^3}dt = \frac{1}{|x|^2} - \frac{1}{R^2}$, we have

$$\begin{aligned}
\int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega} \frac{|\nabla u(x)|^2}{|x|^2} &\leq \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega}\int_{|x|}^{R} \frac{2|\nabla u(x)|^2}{t^3}dtdx + \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega}\frac{|\nabla u|^2}{R^2} \\
&\leq \int_{\sqrt{R}}^{R}\int_{B_t\setminus B_{\sqrt{R}}\cap\Omega}\frac{2|\nabla u(x)|^2}{t^3}dxdt + \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega}\frac{|\nabla u|^2}{R^2} \\
&\leq \int_{\sqrt{R}}^{R}\frac{2}{t^3}\left[\int_{B_t\setminus B_{\sqrt{R}}\cap\Omega}|\nabla u(x)|^2dx\right]dt + \frac{1}{R^2}\int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega}|\nabla u|^2 \\
&= o(\ln^2 R) \qquad as\ \ R\longrightarrow\infty,
\end{aligned},$$

where in the latter line we have used the assumption (2.3.1). Using this information, and letting $R \longrightarrow \infty$ in (2.3.4), we obtain that $\mathcal{K}^2 + |\nabla_T|\nabla u||^2 \equiv 0$ in Ω. The latter implies that Ω is an affine half-space and u is one-dimensional (see the level set analysis in Section 2.1. of [FV10]). This proves the result. $\square$

Theorem 2.3.2. *Let $\Omega\subset\mathbb{R}^2$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3 and let $u\in C^1(\overline{\Omega})\cap C^2(\Omega)$ be a solution to* (2.1.1).

Assume that one of the following hypotheses holds true :

$$f\in Lip_{loc}([0,+\infty)), f(0)\geq 0 \ and\ \nabla u\in L^\infty(\Omega); \tag{H1}$$

$$\begin{cases} f\in Lip_{loc}([0,+\infty)), \quad f(0)\geq 0, \quad \exists\,\zeta>0\,:\,f(t)\leq 0 \quad in \quad [\zeta,+\infty), \\ \exists\,\alpha\in[0,1)\,: \quad u(x)=O(|x|^\alpha) \qquad as \quad |x|\longrightarrow\infty; \end{cases} \tag{H2}$$

$$\begin{cases} f\in Lip([0,+\infty)), \quad f(0)\geq 0, \quad \exists\,\zeta>0\,:\,f(t)\geq 0 \quad in \quad [\zeta,+\infty), \\ u(x)=o(\ln|x|), \qquad as \quad |x|\longrightarrow\infty; \end{cases} \tag{H3}$$

$$\begin{cases} f\in Lip([0,+\infty)), \quad f(0)\geq 0, \quad f(t)\leq 0 \quad in \quad (0,+\infty), \\ u(x)=o(|x|\ln^{\frac{1}{2}}|x|), \qquad as \quad |x|\longrightarrow\infty. \end{cases} \tag{H4}$$

Then, $\Omega=\mathbb{R}^2_+$ up to a vertical translation and there exists $u_0:[0,+\infty)\to(0,+\infty)$ strictly increasing such that

$$u(x)=u_0(x_2)\quad \forall x\in\mathbb{R}^2_+.$$

Proof. Assume (H1). By Corollary 1.5 of [BFS25] u is monotone, i.e., $\frac{\partial u}{\partial x_2} > 0$ in Ω. Furthermore, we have

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 \leq \|\nabla u\|^2_{L^\infty(\Omega)} \pi R^2 \qquad R > 0,$$

The conclusion then follows from Theorem 2.3.1.

Assume (H2). By Proposition 2.5.1 u is bounded. Then, also ∇u is bounded (see Corollary 2.5.2). We can apply the previous step to conclude.

Assume (H3). The growth assumption on u implies that u has at most exponential growth on any finite strip $\Omega \cap \{x_N < R\}$. Therefore, we can apply Theorem 1.3 of [BFS25] to obtain that $\frac{\partial u}{\partial x_2} > 0$ in Ω. To get the desired result, all we need to do is to prove that (2.3.1) holds true. To this end, for $t > 0$ we set $M(t) := \sup_{B(0,t)\cap\Omega} u$ and then we pick a function $\varphi \in C^1_c(\mathbb{R}^2)$ such that $0 \leq \varphi \leq 1$,

$$\varphi(x) := \begin{cases} 1 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 2, \end{cases}$$

and, for any $R > 0$ and $x \in \mathbb{R}^2$ we set $\varphi_R(x) = \varphi(\frac{x}{R})$.

Then we multiply the equation by $(u - M(2R))\varphi_R^2$ and we integrate by parts to find

$$\begin{aligned}
&\int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 + \int_{B_{2R}\cap\Omega} 2\varphi_R(u - M(2R))\nabla u \nabla \varphi_R - \int_{B_{2R}\cap\partial\Omega} \frac{\partial u}{\partial \eta}(u - M(2R))\varphi_R^2 \\
&= \int_{B_{2R}\cap\Omega} f(u)(u - M(2R))\varphi_R^2 \leq \int_{(B_{2R}\cap\Omega)\cap\{u<\zeta\}} f(u)(u - M(2R))\varphi_R^2 \\
&\leq 4\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 \leq& - \int_{B_{2R}\cap\Omega} 2\varphi_R(u - M(2R))\nabla u \nabla \varphi_R \\
&+ |\mathfrak{c}| M(2R) \int_{B_{2R}\cap\partial\Omega} \varphi_R^2 + 4\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2 \\
&\leq \frac{1}{2} \int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 + 2 \int_{B_{2R}\cap\Omega} (u - M(2R))^2 |\nabla \varphi_R|^2 \\
&+ |\mathfrak{c}| M(2R) \mathcal{H}^1(B_{2R} \cap \partial\Omega) + 4\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2
\end{aligned}$$

and so

$$\begin{aligned}
\int_{B_R\cap\Omega} |\nabla u|^2 \leq& 4 \int_{B_{2R}\cap\Omega} (M(2R) - u)^2 |\nabla \varphi_R|^2 \\
&+ 2|\mathfrak{c}| M(2R) \mathcal{H}^1(B_{2R} \cap \partial\Omega) + 8\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2 \\
&\leq 4M^2(2R) \int_{B_{2R}\cap\Omega} |\nabla \varphi_R|^2 + 2|\mathfrak{c}| M(2R) \mathcal{H}^1(B_{2R} \cap \partial\Omega) \\
&+ 8\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2 \leq 4M^2(2R) \|\nabla \varphi\|^2_{L^\infty(\mathbb{R}^2)} R^{-2} \omega_2 (2R)^2 \\
&+ 2|\mathfrak{c}| M(2R) \mathcal{H}^1(B_{2R} \cap \partial\Omega) + 8\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2 \\
&\leq 16\omega_2 \|\nabla \varphi\|^2_{L^\infty(\mathbb{R}^2)} M^2(2R) + 2|\mathfrak{c}| M(2R) \mathcal{H}^1(B_{2R} \cap \partial\Omega) \\
&+ 8\omega_2 \|f\|_{L^\infty([0,\zeta])} M(2R) R^2,
\end{aligned} \tag{2.3.5}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

We also observe that

$$\mathcal{H}^1(B_{2R} \cap \partial\Omega) \leq \int_{(-2R,2R)\subset\mathbb{R}} \sqrt{1+|g'(x_1)|^2}dx_1 \leq 4(1+\|g'\|_{L^\infty(\mathbb{R})})R,$$

since g is globally Lipschitz-continuous by assumption. Then, from (2.3.5) and the growth assumption on u, we deduce that

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \ln R) \qquad as \;\; R \longrightarrow \infty.$$

Assume (H4). In view of the growth assumption on u we can apply Theorem 1.3 of [BFS25] to obtain that $\frac{\partial u}{\partial x_2} > 0$ in Ω. To conclude, we prove that (2.3.1) holds true. To this purpose we multiply the equation by $u\varphi_R^2$ and we integrate by parts to find

$$\int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 + \int_{B_{2R}\cap\Omega} 2\varphi_R u \nabla u \nabla \varphi_R - \int_{B_{2R}\cap\partial\Omega} \frac{\partial u}{\partial \eta} u \varphi_R^2 = \int_{B_{2R}\cap\Omega} f(u) u \varphi_R^2 \leq 0,$$

hence

$$\begin{aligned}
\int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 &\leq - \int_{B_{2R}\cap\Omega} 2\varphi_R u \nabla u \nabla \varphi_R + \int_{B_{2R}\cap\partial\Omega} \frac{\partial u}{\partial \eta} u \varphi_R^2 \\
&= - \int_{B_{2R}\cap\Omega} 2\varphi_R u \nabla u \nabla \varphi_R + 0 \\
&\leq \frac{1}{2} \int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 + 2 \int_{B_{2R}\cap\Omega} u^2 |\nabla \varphi_R|^2.
\end{aligned}$$

Therefore, for any $R > 0$,

$$\int_{B_R\cap\Omega} |\nabla u|^2 \leq \int_{B_{2R}\cap\Omega} |\nabla u|^2 \varphi_R^2 \leq 4 \int_{B_{2R}\cap\Omega} u^2 |\nabla \varphi_R|^2 \leq 16\omega_2 \|\nabla \varphi\|^2_{L^\infty(\mathbb{R}^2)} M^2(2R)$$

and the condition (2.3.1) then follows. □

Theorem 2.3.3. *Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3 and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to* (2.1.1).

Assume that one of the following assumptions holds true :

$$\begin{cases} f \in Lip([0,+\infty)), \quad f(t) \geq 0 \quad for \; any \quad t \geq 0, \\ \\ u(x) = o(\ln|x|), \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{A1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad \exists\, \zeta \geq 0 : f(t) \geq 0 \quad on \quad [0,\zeta] \quad and \quad f(t) \leq 0 \quad on \quad [\zeta,+\infty), \\ \\ \exists\, \alpha \in [0,1) : \quad u(x) = O(|x|^\alpha) \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{A2}$$

Then, $\Omega = \mathbb{R}^3_+$ *up to a vertical translation and there exists* $u_0 : [0, +\infty) \to (0, +\infty)$ *strictly increasing such that*

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Proof. Assume (A1). The proof follows the same lines as the proof of the previous Theorem under the assumption (H3). First, by Theorem 1.3 of [BFS25] we obtain that $\frac{\partial u}{\partial x_3} > 0$ in Ω. Then we multiply the equation by $(u - M(2R))\varphi_R^2$, where now $\varphi \in C_c^3(\mathbb{R}^3)$, and we integrate by parts to find

$$\begin{aligned}&\int_{B_{2R}\cap\Omega} |\nabla u|^2\varphi_R^2 + \int_{B_{2R}\cap\Omega} 2\varphi_R(u - M(2R))\nabla u\nabla\varphi_R - \int_{B_{2R}\cap\partial\Omega} \frac{\partial u}{\partial\eta}(u - M(2R))\varphi_R^2 \\ &= \int_{B_{2R}\cap\Omega} f(u)(u - M(2R))\varphi_R^2 \leq 0,\end{aligned}$$

and so

$$\begin{aligned}\int_{B_R\cap\Omega} |\nabla u|^2 &\leq 4\int_{B_{2R}\cap\Omega} (M(2R) - u)^2|\nabla\varphi_R|^2 + 2|\mathfrak{c}|M(2R)\mathcal{H}^2(B_{2R}\cap\partial\Omega) \\ &\leq 32\omega_3\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^3)}M^2(2R)R + 2|\mathfrak{c}|M(2R)\mathcal{H}^2(B_{2R}\cap\partial\Omega).\end{aligned} \tag{2.3.6}$$

where ω_3 is the measure of the 3-dimensional unit ball and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure.

On the other hand,

$$\mathcal{H}^2(B_{2R}\cap\partial\Omega) \leq \int_{B'(0',2R)\subset\mathbb{R}^2} \sqrt{1 + |\nabla' g(x')|^2}dx' \leq 4\pi(1 + \|\nabla' g\|_{L^\infty(\mathbb{R}^2)})R^2,$$

since g is globally Lipschitz-continuous by assumption. Consequently, for any $R > 0$, we have

$$\begin{aligned}\int_{B_R\cap\Omega} |\nabla u|^2 &\leq 32\omega_3\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^3)}M^2(2R)R + 2|\mathfrak{c}|M(2R)\mathcal{H}^2(B_{2R}\cap\partial\Omega) \\ &\leq 32\omega_3\|\nabla\varphi\|^2_{L^\infty(\mathbb{R}^3)}M^2(2R)R + 8|\mathfrak{c}|\pi(1 + \|\nabla' g\|_{L^\infty(\mathbb{R}^2)})M(2R)R^2.\end{aligned}$$

The latter and the assumption on u imply the validity of the condition (2.3.1). Then, the conclusion follows by applying Theorem 2.3.1.

Assume (A2). By Proposition 2.5.1 u is bounded above by ζ. Hence, u is a bounded solution to

$$\begin{cases} -\Delta u = \tilde{f}(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial\eta} = \mathfrak{c} = const. & \text{on} \quad \partial\Omega, \end{cases}$$

where $\tilde{f}$ is the non-negative globally Lipschitz-continuous function defined by $\tilde{f} = f$ in $[0, \zeta]$ and $\tilde{f} \equiv f(\zeta) = 0$ in $[\zeta, +\infty)$. Therefore, we can apply the previous step to conclude. □

When f is of class C^1 we can extend the above result to a larger class of nonlinear functions f, with $f(0) \geq 0$. Specifically, we have

Theorem 2.3.4. *Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz-continuous epigraph bounded from below and with boundary of class C^3.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a bounded solution to (2.1.1), where $f \in C^1([0,+\infty))$ satisfies $f(0) \geq 0$ and

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t), \tag{2.3.7}$$

here F denotes the primitive of f vanishing at 0.[4]

Then, $\Omega = \mathbb{R}^3_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

Remark 2.3.2. Note that the assumption (2.3.7) is natural and somehow necessary. Indeed, if the *Conjecture BCNe* is true, i.e., if Ω is an affine half-space, then the Unique Continuation Principle ensures that the solution u must be one-dimensional. Next, a standard analysis of the first integral gives that $F(\sup u) > F(t)$, for any $t \in [0, \sup u)$, thus confirming the validity of (2.3.7).

The proof of Theorem 2.3.4 is based on the following general rigidity result for monotone solutions to the overdetermined problem (2.1.1) on 3-dimensional epigraphs.

Theorem 2.3.5. *Let $u \in C^2(\overline{\Omega})$ be a bounded solution to (2.1.1) where $\Omega \subset \mathbb{R}^3$ is an epigraph with boundary of class C^3 such that*

$$\mathcal{H}^2(\partial\Omega \cap B(0,R)) = o(R^2 \ln(R)) \qquad \textit{as } R \longrightarrow \infty, \tag{2.3.8}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure.

Let $f \in C^1([0,+\infty))$ be such that

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t). \tag{2.3.9}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_3}(x) > 0 \qquad \forall x \in \Omega, \tag{2.3.10}$$

then, $\Omega = \mathbb{R}^3_+$ up to isometry and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

[4] $F(t) = \displaystyle\int_0^t f(s)ds, \qquad t \geq 0.$

Remark 2.3.3. Note explicitly that in the above Theorem :

(i) no restriction is imposed on the sign of $f(0)$ or on $\mathfrak{c}$;

(ii) the epigraph Ω is not assumed to be bounded from below.

Proof of Theorem 2.3.5. From Lemma 2.5.2 and assumption (2.3.8) we have

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 = o(R^2 \log R) \qquad as \;\; R \longrightarrow \infty. \tag{2.3.11}$$

The conclusion then follows by appyling Theorem 2.3.1. $\square$

Proof of Theorem 2.3.4. Under the assumptions of Theorem 2.3.4 we can apply Theorem 1.3 of [BFS25] to get that $\frac{\partial u}{\partial x_3} > 0$ on Ω. Moreover

$$\mathcal{H}^2(B_R \cap \partial\Omega) \leq \int_{B'(0',R)\subset\mathbb{R}^2} \sqrt{1+|\nabla' g(x')|^2}dx' \leq \pi(1+\|\nabla' g\|_{L^\infty(\mathbb{R}^2)})R^2,$$

since g is globally Lipschitz-continuous by assumption. Hence, assumption (2.3.8) is satisfied. We conclude by applying Theorem 2.3.5. $\square$

2.4 Serrin's overdetermined problem with general nonlinearity. The case $f(0) < 0$.

In this section we consider the case of locally Lipschitz-continuous functions f with $f(0) < 0$. We first prove a new monotonicity result in any dimension and then we deduce the validity of the *conjecture BCNe* for $N \leq 3$, for possibly unbounded solutions. We explicitely note that the results of this section are new, even in the case of bounded solutions.

2.4.1 A new monotonicity result in any dimension $N \geq 2$

In order to state our results we need the following

Definition 2.4.1. Assume $N \geq 2$ and $\gamma \in (0,1]$. We denote by $\mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ the set of functions $h \in C^1(\mathbb{R}^{N-1})$ such that ∇h is bounded and globally γ-Hölder continuous, i.e., such that

$$\begin{aligned}\|\nabla h\|_{C^{0,\gamma}(\mathbb{R}^{N-1})} &:= \|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})} + [\nabla h]_{C^{0,\gamma}(\mathbb{R}^{N-1})} \\ &:= \sup_{x\in\mathbb{R}^{N-1}} |\nabla h(x)| + \sup_{x,y\in\mathbb{R}^{N-1}, x\neq y} \frac{|\nabla h(x) - \nabla h(y)|}{|x-y|^\gamma} < +\infty.\end{aligned}$$

Notice that, we are not assuming that h is bounded on $\mathbb{R}^{N-1}$. However, any member of $\mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ is globally Lipschitz-continuous on $\mathbb{R}^{N-1}$, since ∇h is supposed to be bounded on $\mathbb{R}^{N-1}$.

Theorem 2.4.2. *Assume $N \geq 2$, $\gamma \in (0,1]$. Let Ω be an epigraph bounded from below and defined by a function $g \in \mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ and let $f \in Lip_{loc}(\mathbb{R})$ with $f(0) < 0$.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (2.1.1) with $\mathfrak{c} \neq 0$ and such that ∇u is bounded on finite strips.[5]
Then, u is monotone, i.e.

$$\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega.$$

Remark 2.4.1. In particular, the above Theorem applies to bounded solutions to (2.1.1). Indeed, bounded solutions to (2.1.1) also have bounded gradient (see Corollary 2.5.2).

Proof. The proof is inspired by those in our recent paper [BFS25] (see Section 4 therein). Thanks to the translation invariance of problem (2.1.1), we may and do suppose that $\inf_{\mathcal{R}^\nu} g = 0$. Moreover, by the assumption on ∇u, it is immediate to see that u is bounded and uniformly continuous on any finite strip $\Omega \cap \{x_N < R\}$, $R > 0$.

Therefore, as in [BFS25], we can use a variant of the moving planes method *suitably adapted to the geometry of the epigraph.* To this end, let us recall the notations used in [BFS25].

For $0 < a < b$ and $\lambda > 0$ we set:

$$\Sigma_\lambda := \{\, x \in \mathbb{R}^N \,:\, 0 < x_N < \lambda \,\},$$
$$\Sigma_b^g = \{x = (x', x_N) \in \mathbb{R}^N \,:\, g(x') < x_N < b\},$$
$$\Sigma_{a,b}^g = \{x = (x', x_N) \in \mathbb{R}^N \,:\, g(x') + a < x_N < b\},$$

$$\forall x \in \Sigma_\lambda^g, \qquad u_\lambda(x) = u(x', 2\lambda - x_N).$$

Also, for any subset $S \subseteq \mathbb{R}^N$, we denote by $UC(S)$ the set of uniformly continuous functions defined on S.

We proceed as in the proof of Theorem 1.1 of [BFS25].

We set $\Lambda := \{t > 0 \,:\, u \leqslant u_\theta \text{ in } \Sigma_\theta^g,\ \forall\, 0 < \theta < t\}$ and we aim at proving that

$$\tilde{t} := \sup \Lambda = +\infty.$$

Exactly as in the first step of the proof of Theorem 1.1 of [BFS25] we have :

[5] i.e., for any $R > 0$,

$$\sup_{\Omega \cap \{x_N < R\}} |\nabla u| < +\infty.$$

Step 1 : Λ is not empty.

We note explicitely that this result holds true irrespectively of the value of $f(0)$ (see also Corollary 5.5 of [BFS25]).

Step 2 : $\tilde{t} = \sup \Lambda = +\infty$.

To this aim, we establish the following

Proposition 2.4.1. *For every $\delta \in (0, \frac{\tilde{t}}{2})$, there is $\varepsilon(\delta) > 0$ such that*

$$\forall \varepsilon \in (0, \varepsilon(\delta)) \quad u \leq u_{\tilde{t}+\varepsilon} \quad \text{on} \quad \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}.$$

Proof of Proposition 2.4.1. Suppose that there exists $\delta \in \left(0, \frac{\tilde{t}}{2}\right)$ such that

$$\forall k \geq 1 \quad \exists \varepsilon_k \in \left(0, \frac{1}{k}\right) \quad \exists x^k \in \overline{\Sigma^g_{\delta,\tilde{t}-\delta}} \quad : \quad u(x^k) \geq u_{\tilde{t}+\varepsilon_k}(x^k). \tag{2.4.1}$$

Since $(x^k)_{k\in\mathbb{N}} \in \overline{\Sigma^g_{\delta,\tilde{t}-\delta}}$, we have

$$\delta < g((x^k)') + \delta \leq x^k_N \leq \tilde{t} - \delta, \quad \forall k \geq 1, \tag{2.4.2}$$

so the sequence (x^k_N) is bounded. Thus there exists $x^\infty_N \in [\delta, \tilde{t} - \delta]$ such that, up to a subsequence, $x^k_N \to x^\infty_N$ as $k \to +\infty$.
Now we set

$$g_k(x') = g((x^k)' + x'), \quad \forall x' \in \mathcal{R}^\nu \quad \forall k \geq 1. \tag{2.4.3}$$

Thanks to (2.4.2) we have $0 \leq g_k(0') = g((x^k)') \leq \tilde{t}$, for any $k \geq 1$. The latter and the assumption $g \in \mathbb{C}^{1,\gamma}(\mathbb{R}^{N-1})$ immediately imply that the sequence (g_k) is bounded in $C^{1,\gamma}(\mathcal{K})$, for any compact set $\mathcal{K} \subset \mathbb{R}^{N-1}$. Hence, there exists $g_\infty \in C^{1,\gamma}_{loc}(\mathbb{R}^{N-1})$ such that, up to a subsequence, $g_k \longrightarrow g_\infty$ in $C^1_{\text{loc}}(\mathbb{R}^{N-1})$.

We also observe that, passing to the limit in (2.4.2), we obtain

$$\delta \leq g_\infty(0') + \delta \leq x^\infty_N \leq \tilde{t} - \delta. \tag{2.4.4}$$

For any $k \geq 1$, let us consider $\Omega^k = \{(x', x_N) \in \mathbb{R}^N,\, x_N > g_k(x')\}$, the epigraph of g_k, $\Omega^\infty = \{(x', x_N) \in \mathbb{R}^N,\, x_N > g_\infty(x')\}$, the epigraph of g_∞ and define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \overline{\Omega}, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{2.4.5}$$

Clearly, $\tilde{u}$ is bounded and uniformly continuous on any finite strip $\{x_N < R\}$ so, the sequence $(\tilde{u}_k)$ defined by

$$\tilde{u}_k(x) = \tilde{u}(x' + (x^k)', x_N) \qquad \forall\, x = (x', x_N) \in \mathbb{R}^N, \quad \forall k \geq 1, \tag{2.4.6}$$

is relatively compact in $C^0_{\mathrm{loc}}(\mathbb{R}^N)$ (by Ascoli-Arzelà theorem).
Therefore, there exists $\tilde{u}_\infty \in C^0(\mathbb{R}^N)$ such that, up to a subsequence, $\tilde{u}_k \to \tilde{u}_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\begin{cases} -\Delta \tilde{u}_\infty = f(\tilde{u}_\infty) & \text{in} \quad \mathcal{D}'(\Omega^\infty), \\ \tilde{u}_\infty \geq 0 & \text{in} \quad \Omega^\infty, \\ \tilde{u}_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \end{cases} \tag{2.4.7}$$

where the boundary condition follows by observing that,

$$\forall\, k \geq 1, \quad \forall\, x' \in \mathbb{R}^{N-1} \qquad 0 = \tilde{u}_k(x', g_k(x')),$$

and thus

$$0 = \tilde{u}_k(x', g_k(x')) \longrightarrow \tilde{u}_\infty(x', g_\infty(x')) \qquad \text{as } k \to \infty,$$

thanks to the uniform convergence of $(\tilde{u}_k)$ and (g_k) on compact sets.

By construction, $u_\infty := \tilde{u}_{\infty|\Omega^\infty} \in C^0(\overline{\Omega^\infty})$. Furthermore, $u_\infty \in C^2(\Omega^\infty)$ by (2.4.7) and standard interior regularity theory for elliptic equations, and it satisfies

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ u_\infty \geq 0 & \text{in} \quad \Omega^\infty, \\ u_\infty = 0 & \text{on} \quad \partial\Omega^\infty. \end{cases} \tag{2.4.8}$$

Now we deal with the Neumann boundary condition.

Since $g_\infty(0') < \tilde{t} - 2\delta$, by the continuity of g_∞ and (2.4.2), we can choose $R > 0$ small enough such that

$$\sup_{x' \in \overline{B'(0', 4R)}} g_\infty(x') < \tilde{t} - \delta, \tag{2.4.9}$$

where $B'(0', 4R) \subset \mathbb{R}^{N-1}$. Since $g_k \to g_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R}^{N-1})$, there exists $k_1 = k_1(R) \geq 1$ such that

$$\forall k \geq k_1, \quad \forall x' \in \overline{B'(0', 4R)}, \qquad g_\infty(x') - \frac{\delta}{4} \leq g_k(x') \leq g_\infty(x') + \frac{\delta}{4}. \tag{2.4.10}$$

Pick $T > 2\tilde{t} + 2R$ and for any $k \geq k_1$, we define on $\overline{\Omega^k}$

$$u_k(x) = u(x' + (x^k)', x_N).$$

Since u solves (2.1.1), then $u_k \in C^{2,\gamma}(\overline{\mathfrak{C}^{g_k}(0', 2R, 2T)})$ and

$$\begin{cases} -\Delta u_k = f(u_k) & \text{in} \qquad \mathfrak{C}^{g_k}(0', 2R, 2T), \\ u_k \geq 0 & \text{in} \qquad \mathfrak{C}^{g_k}(0', 2R, 2T), \\ u_k = 0 & \text{on} \quad \partial\Omega^k \cap \widehat{\mathfrak{C}^{g_k}}(0', 2R, 2T), \\ \dfrac{\partial u_k}{\partial \eta_k} = \mathfrak{c} & \text{on} \quad \partial\Omega^k \cap \widehat{\mathfrak{C}^{g_k}}(0', 2R, 2T), \end{cases} \tag{2.4.11}$$

where $\mathfrak{C}^{g_k}(0', r, \tau) = \Big\{x = (x', x_N) \in \mathbb{R}^N,\ x' \in B'(0', r) \ \text{ and } \ g_k(x') < x_N < \tau\Big\}$ and η_k denotes the outward unit normal vector to $\partial\Omega^k$.

At this point, we apply Proposition 3.1. of [BFS25] with $u = u_k$, $\widetilde{R} = 2R$, $\widetilde{\tau} = 2T$, $r = \frac{3R}{4}$ and $t = \frac{5T}{4}$, to get the existence of constants $\alpha_0 = \alpha_0(N, g) \in (0, 1)$ and $C_0 = C_0(R, T, N, g) > 0$ such that, for any $k \geq k_1$,

$$\|u_k\|_{C^{0,\alpha_0}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \leq C_0(\|f'\|_{L^\infty([0,M_k])} + |f(0)| + 1)(\|u_k\|_{L^\infty(\Omega\cap\{x_N<2T\})} + 1), \tag{2.4.12}$$

where $M_k = \sup\limits_{\mathfrak{C}^{g_k}(0',2R,2T)} u_k$.

Since $M_k \leq \|u\|_{L^\infty(\Omega\cap\{x_N<2T\})}$, from the latter we infer that, for any $k \geq k_1$,

$$\|u_k\|_{C^{0,\alpha_0}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \leq C_0(\|f'\|_{L^\infty([0,M])} + |f(0)| + 1)(M + 1), \tag{2.4.13}$$

where $M := \|u\|_{L^\infty(\Omega\cap\{x_N<2T\})}$.

Now we apply Theorem 2.5.2 with $u = u_k$, $\widetilde{R} = 2R$, $\widetilde{\tau} = 2T$, $r = \frac{3R}{4}$ and $t = \frac{5T}{4}$, to deduce the existence of constants $\alpha_1 = \alpha_1(N, g) \in (0, 1)$ and $C_1 = C_1(R, T, N, \gamma) > 0$ such that, for any $k \geq k_1$,

$$\begin{aligned}
&\|\nabla u_k\|_{C^{0,\alpha_1}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \leq \|\nabla u_k\|_{L^\infty(\Omega\cap\{x_N<2T\})} + [\nabla u_k]_{C^{0,\alpha_1}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \\
&\leq \|\nabla u\|_{L^\infty(\Omega\cap\{x_N<2T\})} \\
&+ C_1\left[|\mathfrak{c}|\left(1 + \|\nabla g_k\|_{C^{0,\gamma}(\mathbb{R}^{N-1})}\right)^2 + 1 + \|f'\|_{L^\infty([0,M_k])}\right](1 + \|\nabla u_k\|_{L^\infty(\mathfrak{C}^{g_k}(0',2R,2T))}) \\
&\leq \|\nabla u\|_{L^\infty(\Omega\cap\{x_N<2T\})} \\
&+ C_1\left[|\mathfrak{c}|\left(1 + \|\nabla g\|_{C^{0,\gamma}(\mathbb{R}^{N-1})}\right)^2 + 1 + \|f'\|_{L^\infty([0,M])}\right](1 + \|\nabla u\|_{L^\infty(\Omega\cap\{x_N<2T\})})
\end{aligned} \tag{2.4.14}$$

Hence, by combining (2.4.13) and (2.4.14) we can find a constant C', independent of k, such that

$$\|u_k\|_{C^{1,\alpha}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \leq C' \qquad \forall\, k \geq k_1, \tag{2.4.15}$$

where $\alpha = \min\{\alpha_0, \alpha_1, \gamma\} \in (0, 1)$.

Set $\mathcal{K} := \overline{B'(0', R/2) \times (-T, T)}$. We can then apply item (ii) of Lemma 2.5.6 (with $r = 3R/4$, $\widetilde{T} = 5T/4$ and $t = T$) to see that any u_k has an extension $v_k \in C^{1,\alpha}(\mathcal{K})$ satisfying

$$\|v_k\|_{C^{1,\alpha}(\mathcal{K})} \leq C_{ext}(1 + \|\nabla g_k\|_{C^{0,\alpha}(\mathbb{R}^{N-1})})^4 \|u_k\|_{C^{1,\alpha}(\overline{\mathfrak{C}^{g_k}(0',3R/4,5T/4)})} \qquad \forall\, k \geq k_1, \tag{2.4.16}$$

where C_{ext} is a positive constant depending only on N and T. Hence

$$\|v_k\|_{C^{1,\alpha}(\mathcal{K})} \leq C' C_{ext}(1 + \|\nabla g\|_{C^{0,\alpha}(\mathbb{R}^{N-1})})^4 := C'' \qquad \forall\, k \geq k_1, \tag{2.4.17}$$

where C'' is a constant independent of k.

Thus, by Ascoli-Arzelà Theorem, there exists $v_\infty \in C^{1,\alpha}(\mathcal{K})$ such that, up to a subsequence,

$$v_k \longrightarrow v_\infty \quad \text{in } C^1(\mathcal{K}). \tag{2.4.18}$$

Now we are ready to prove that $\dfrac{\partial u_\infty}{\partial \eta_\infty} = \mathfrak{c}$ on $\mathcal{K} \cap \partial\Omega^\infty$, where η_∞ denotes the outward unit normal vector to $\partial\Omega^\infty$ (recall that $g_\infty \in C^1(\mathbb{R}^{N-1})$ by construction).

Observe that, for any compact set $K \subset \mathbb{R}^{N-1}$ and any $x' \in K$, we have

$$\begin{aligned} |\eta_k(x', g_k(x')) - \eta_\infty(x', g_\infty(x'))| &= \Big| \frac{(\nabla g_k(x'), -1)^T}{\sqrt{1 + |\nabla g_k(x')|^2}} - \frac{(\nabla g_\infty(x'), -1)^T}{\sqrt{1 + |\nabla g_\infty(x')|^2}} \Big| \\ &\le 2|\nabla g_k(x') - \nabla g_\infty(x')| \longrightarrow 0, \quad \text{as} \quad k \longrightarrow \infty \end{aligned} \tag{2.4.19}$$

since $g_k \longrightarrow g_\infty$ in $C^1_{loc}(\mathbb{R}^{N-1})$.

Hence, for any $y \in \mathcal{K} \cap \partial\Omega^\infty$ we get

$$\nabla v_\infty(y) \cdot \eta_\infty(y', g_\infty(y')) = \lim_{k \to \infty} \nabla v_k(y) \cdot \eta_k(y', g_k(y')). \tag{2.4.20}$$

Also, for any $k \ge k_1$ and any $y \in \mathcal{K} \cap \partial\Omega^\infty$ we have

$$\begin{aligned} &\nabla v_k(y) \cdot \eta_k(y', g_k(y')) = \nabla v_k(y) \cdot \eta_k(y', g_k(y')) - \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y')) \\ &+ \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y')) = (\nabla v_k(y) \cdot \eta_k(y', g_k(y')) - \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y'))) \\ &+ \frac{\partial v_k}{\partial \eta_k}(y', g_k(y')) = (\nabla v_k(y) \cdot \eta_k(y', g_k(y')) - \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y'))) \\ &+ \frac{\partial u_k}{\partial \eta_k}(y', g_k(y')) = (\nabla v_k(y) \cdot \eta_k(y', g_k(y')) - \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y'))) + \mathfrak{c}, \end{aligned} \tag{2.4.21}$$

where in the latter line we have used that $v_{k|\mathcal{K} \cap \overline{\Omega^k}} = u_k$ and (2.4.11).

To proceed further, we note that

$$\begin{aligned} &|\nabla v_k(y) \cdot \eta_k(y', g_k(y')) - \nabla v_k(y', g_k(y')) \cdot \eta_k(y', g_k(y'))| \le |\nabla v_k(y) - \nabla v_k(y', g_k(y'))| \\ &\le C''|(y', g_\infty(y')) - (y', g_k(y'))|^\alpha = C''|g_\infty(y') - g_k(y'))|^\alpha \longrightarrow 0, \quad \text{as} \quad k \longrightarrow \infty, \end{aligned} \tag{2.4.22}$$

where in the latter line we have used (2.4.17) and that $g_k \longrightarrow g_\infty$ in $C^1_{loc}(\mathbb{R}^{N-1})$.

By putting together (2.4.20)-(2.4.22) and recalling that $v_{\infty|\mathcal{K} \cap \overline{\Omega^\infty}} = u_\infty$ we obtain

$$\frac{\partial u_\infty}{\partial \eta_\infty}(y) = \frac{\partial v_\infty}{\partial \eta_\infty}(y) = \mathfrak{c}, \qquad \forall\, y \in \mathcal{K} \cap \partial\Omega^\infty. \tag{2.4.23}$$

By summarizing we have that $u_\infty \in C^0(\overline{\Omega^\infty}) \cap C^2(\Omega^\infty) \cap C^{1,\alpha}(\mathcal{K} \cap \overline{\Omega^\infty})$ is a classical solution to

$$\begin{cases} -\Delta u_\infty = f(u_\infty) & \text{in} \quad \Omega^\infty, \\ \quad u_\infty \ge 0 & \text{in} \quad \Omega^\infty, \\ \quad u_\infty = 0 & \text{on} \quad \partial\Omega^\infty, \\ \quad \dfrac{\partial u_\infty}{\partial \eta_\infty} = c & \text{on} \quad \mathcal{K} \cap \partial\Omega^\infty. \end{cases} \tag{2.4.24}$$

Now, by definition of $\tilde{t}$, we have

$$u_k(x) \leq u_{k,\tilde{t}}(x) \qquad \forall x \in \overline{\mathfrak{C}^{g_k}(0', R/4, \tilde{t})} \subset \mathcal{K} \cap \overline{\Omega^\infty}, \quad \forall\, k \geq k_1. \tag{2.4.25}$$

By letting $k \longrightarrow +\infty$ in (2.4.25) we deduce that

$$u_\infty(x) \leq u_{\infty,\tilde{t}}(x) \qquad \forall x \in \overline{\mathfrak{C}^{g_\infty}(0', R/4, \tilde{t})}. \tag{2.4.26}$$

We notice that (2.4.1) implies $u_k(0', x_N^k) > u_{k,\tilde{t}+\varepsilon_k}(0', x_N^k)$, for any $k \geq 1$. Then, passing to the limit in the latter we deduce

$$u_\infty(0', x_N^\infty) \geq u_{\infty,\tilde{t}}(0', x_N^\infty),$$

and so

$$u_\infty(0', x_N^\infty) = u_{\infty,\tilde{t}}(0', x_N^\infty)$$

since $(0', x_N^\infty) \in \mathfrak{C}^{g_\infty}(0', R/4, \tilde{t})$ and (2.4.26) holds true.

Now, for any $x \in \mathfrak{C}^{g_\infty}(0', R/4, \tilde{t})$ we set

$$w_\infty(x) = u_{\infty,\tilde{t}}(x) - u_\infty(x),$$

then w_∞ satisfies

$$\begin{cases} -\Delta w_\infty = f(u_{\infty,\tilde{t}}) - f(u_\infty) \geq -L_f w_\infty & \text{in} \quad \mathfrak{C}^{g_\infty}(0', R/4, \tilde{t}), \\ \qquad w_\infty \geq 0 & \text{in} \quad \mathfrak{C}^{g_\infty}(0', R/4, \tilde{t}), \\ (0', x_N^\infty) \in \mathfrak{C}^{g_\infty}(0', R/4, \tilde{t}), & \qquad w_\infty(0', x_N^\infty) = 0, \end{cases} \tag{2.4.27}$$

where L_f is the Lipschitz constant of f on the compact set $[0, \sup_{\overline{\mathfrak{C}^{g_\infty}(0',R/4,\tilde{t})}} u_{\infty,\tilde{t}}]$.
By the strong maximum principle we infer that $w_\infty \equiv 0$ on the set $\overline{\mathfrak{C}^{g_\infty}(0', R/4, \tilde{t})})$, which in turn gives

$$u_\infty(0', 2\tilde{t} - g_\infty(0')) = u_{\infty,\tilde{t}}(0', g_\infty(0')) = u_\infty(0', g_\infty(0')) = 0. \tag{2.4.28}$$

The latter shows that u_∞ attains its minimum value in Ω^∞ at the point $(0', 2\tilde{t} - g_\infty(0'))$, hence $\nabla u_\infty(0', 2\tilde{t} - g_\infty(0')) = 0$.

Moreover, since $w_\infty \equiv 0$ on the set $\overline{\mathfrak{C}^{g_\infty}(0', R/4, \tilde{t})}$, we also have

$$\nabla u_\infty(0', 2\tilde{t} - g_\infty(0')) = \nabla u_\infty(0', g_\infty(0')) = 0,$$

and so

$$0 = |\nabla u_\infty(0', 2\tilde{t} - g_\infty(0'))| = |\nabla u_\infty(0', g_\infty(0'))| = |\mathfrak{c}| \neq 0,$$

a contradiction. This proves Proposition 2.4.1. □

With Proposition 2.4.1 at our disposal, the conclusion of the proof of *Step 2* is the same as that of the proof of Theorem 1.1 of [BFS25].

Finally, the monotonicity of u is obtained (via the Hopf's Lemma) exactly as in *Step 3* of the proof of Theorem 1.1 of [BFS25]. □

2.4.2 The *conjecture BCNe* when $f(0) < 0$

With the monotonicity result established in Theorem 2.4.2, it is immediate to extend Theorem 2.3.2 and Theorem 2.3.4 to the case $f(0) < 0$. Specifically we have

Theorem 2.4.3. *Let $\Omega \subset \mathbb{R}^2$ be a globally Lipschitz-continuous epigraph bounded from below and defined by a function $g \in C^3(\mathbb{R}) \cap \mathbb{C}^{1,\gamma}(\mathbb{R})$.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (2.1.1) with $\mathfrak{c} \neq 0$ and assume that one of the following hypotheses holds true :

$$f \in Lip_{loc}([0,+\infty)), f(0) < 0 \text{ and } \nabla u \in L^\infty(\Omega); \tag{H1}$$

$$\begin{cases} f \in Lip_{loc}([0,+\infty)), \quad f(0) < 0, \quad \exists\, \zeta > 0 \, : \, f(t) \leq 0 \quad in \quad [\zeta,+\infty), \\ \\ \exists\, \alpha \in [0,1) \, : \quad u(x) = O(|x|^\alpha) \qquad as \quad |x| \longrightarrow \infty; \end{cases} \tag{H2}$$

Then, $\Omega = \mathbb{R}^2_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_2) \quad \forall x \in \mathbb{R}^2_+.$$

and

Theorem 2.4.4. *Let $\Omega \subset \mathbb{R}^3$ be a globally Lipschitz-continuous epigraph bounded from below and defined by a function $g \in C^3(\mathbb{R}^2) \cap \mathbb{C}^{1,\gamma}(\mathbb{R}^2)$.*
Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a bounded solution to (2.1.1) with $\mathfrak{c} \neq 0$.
Let $f \in C^1([0,+\infty))$ be such that $f(0) < 0$ and

$$F(\sup u) = \sup_{t \in [0, \sup u]} F(t). \tag{2.4.29}$$

Then, $\Omega = \mathbb{R}^3_+$ up to a vertical translation and there exists $u_0 : [0,+\infty) \to (0,+\infty)$ strictly increasing such that

$$u(x) = u_0(x_3) \quad \forall x \in \mathbb{R}^3_+.$$

2.5 Some auxiliary results

This section is devoted to the demonstration of several results used to prove our main conclusions. Some of them are general results of independent interest. They could be useful for further studies of the nonlinear Poisson equation on general domains.

2.5.1 Some uniform bounds

Proposition 2.5.1. *Assume $N \geq 2$ and $\alpha \in [0,1)$. Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain contained in an open affine half-space. Let $f \in C^0(\mathbb{R})$ be a function satisfying*

$$\exists\, \zeta > 0 \quad : \quad f(t) \leq 0 \quad on \quad [\zeta,+\infty). \tag{2.5.1}$$

Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to

$$\begin{cases} -\Delta u \leq f(u) & \textit{in} \quad \Omega, \\ \quad u \leq 0 & \textit{on} \quad \partial\Omega, \end{cases}$$

such that, $u^+(x) = O(|x|^\alpha)$ as $|x| \longrightarrow \infty$.

Then

1) *$u \leq \zeta$ on Ω;*

2) *$f(\sup_\Omega u) \geq 0$. Moreover, if $f(\sup_\Omega u) = 0$, then for any $\mu < \sup_\Omega u$ there is $t = t(\mu) \in (\mu, \sup_\Omega u)$ such that $f(t) > 0$.*

Remark 2.5.1.
(i) The assumption $\alpha < 1$ is sharp in the above result. Indeed, $u(x) = x_N$ is a positive unbounded harmonic function on the half-space $\mathbb{R}^N_+$ with zero Dirichlet boundary condition.

(ii) Proposition 2.5.1 also shows that the only bounded solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ \quad u \geq 0 & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

must be $u \equiv 0$.

Actually, the Remark following Lemma 2.1 in [BCN97b] implies that this conclusion holds true for any non dense open set of $\mathbb{R}^2$. In particular, this result holds for any two-dimensional epigraph.

Proof. 1) We argue by contradiction. Assume that there exists $x_0 \in \Omega$ such that $u(x_0) > \zeta$ and let $D \subset \Omega$ be the open connected component of the open set $\{u > \zeta\}$ which contains the point x_0. Then, $u(x) > \zeta$ for any $x \in D$ and $u(x) = \zeta$ for any $x \in \partial D$ (since $\partial D \subset \Omega$ thanks to the boundary condition satisfied by u and the definition of D).

Let $w = \zeta - u$. Then $w < 0$ in D and $w(x) = O(|x|^\alpha)$, when $x \in D$ and $|x| \longrightarrow \infty$. Moreover,

$$\Delta w = -\Delta u \leq f(u) \leq 0 \quad \text{in} \quad D.$$

Then, the Remark[6] following Lemma 2.1 in [BCN97b] yields $w \geq 0$ in D, a contradiction. Therefore, $u \leq \zeta$ on Ω.

If $f(\sup_\Omega u) < 0$, then there is $\varepsilon \in (0, \sup_\Omega u)$ such that $f(t) < 0$ in $[\sup_\Omega u - \varepsilon, \sup_\Omega u]$. Hence, u is a bounded solution to

$$\begin{cases} -\Delta u \leq \tilde{f}(u) & \text{in} \quad \Omega, \\ \quad u \leq 0 & \text{in} \quad \partial\Omega, \end{cases}$$

[6] Indeed, since Ω is contained in an affine half-space, the exponent α, which appears in the barrier g defined by equation (2.1) on p. 1094 of [BCN97b], can be chosen arbitrarily in the open interval $(0, 1)$.

where $\tilde{f} \in C^0(\mathbb{R})$ is defined by $\tilde{f} = f$ in $(-\infty, \sup_\Omega u]$ and $\tilde{f} \equiv f(\sup_\Omega u) < 0$ in $[\sup_\Omega u, +\infty)$. We can then apply item 1) to conclude that $u \le \sup_\Omega u - \varepsilon$ in Ω. A contradiction. The same argument proves the last claim. $\square$

Proposition 2.5.2. *Let $\Omega \subset \mathbb{R}^N$ be a proper domain. Let $h \in C^0([0, +\infty))$ be a function satisfying the Keller-Osserman conditions*

$$\begin{cases} h(0) = 0, \quad h(t) > 0 \quad \textit{for} \quad t > 0, \quad h \textit{ non-decreasing}, \\ \\ \int^{\infty} \left(\int_0^t h(s)ds \right)^{-\frac{1}{2}} < \infty. \end{cases} \tag{2.5.2}$$

Let $v \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a solution to

$$\begin{cases} \Delta v \ge h(v) & \textit{in} \quad \mathcal{D}'(\Omega), \\ v \ge 0 & \textit{on} \quad \Omega, \\ v = 0 & \textit{on} \quad \partial\Omega. \end{cases}$$

Then, $v \equiv 0$ in Ω.

Proof. We recall that, the Keller-Osserman condition (2.5.2) implies that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\Delta w = h(w) \quad \text{in} \quad B(0, R_\varepsilon), \tag{2.5.3}$$

admits a C^2-radially symmetric, positive solution w_ε, which blows up at the boundary of the ball. Furthermore, $w_\varepsilon(0) = \varepsilon = \inf_{B(0,R_\varepsilon)} w_\varepsilon$.

For contradiction, suppose that $v \not\equiv 0$ in Ω. Then, there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. Let $\varepsilon = \frac{v(x_0)}{2} > 0$ and consider the corresponding solution w_ε of (2.5.3), satisfying the above properties. Therefore, the function $u(x) = w_\varepsilon(|x - x_0|)$ is a C^2-radially symmetric (with respect to x_0), positive solution to $\Delta w = h(w)$ in $B(x_0, R_\varepsilon)$, which blows up at the boundary of the ball $B(x_0, R_\varepsilon)$.

Let us consider the open set $U := B(x_0, R_\varepsilon) \cap \Omega$ and set $M := \sup_{\overline{U}} v$. Then, M is finite, since $v \in C^0(\overline{\Omega})$ by assumption. Thus, there is $0 < R(M) < R_\varepsilon$ such that $u(x) > M$ on $B(x_0, R_\varepsilon) \setminus B(x_0, R(M))$ (recall that u blows up at the boundary of the ball $B(x_0, R_\varepsilon)$). The latter and the homogeneous Dirichlet condition satisfied by v, imply that $v - u \le 0$ on $\partial\,[B(x_0, R(M)) \cap \Omega]$. It follows that $(v - u)^+ \in H^1_0(B(x_0, R(M)) \cap \Omega)$ and so, we can take it as test function in the weak formulation of the equation satisfied by $v - u$ in $B(x_0, R(M)) \cap \Omega$, that is, in the weak formulation of $-\Delta(v - u) \le h(u) - h(v)$ in $B(x_0, R(M)) \cap \Omega$. This choice leads to

$$\int |\nabla(v-u)^+|^2 = \int \nabla(v-u)\nabla(v-u)^+ \le \int (h(u) - h(v))(v-u)^+ \le 0$$

since h is non-decreasing. Consequently, $v \le u$ on $B(x_0, R(M)) \cap \Omega$, by the continuity of $(v - u)^+$. In particular we have that $0 < v(x_0) \le u(x_0) = w_\varepsilon(0) = \varepsilon = \frac{v(x_0)}{2}$. A contradiction. This proves the Proposition. $\square$

Corollary 2.5.1. *Let $\Omega \subset \mathbb{R}^N$ be a proper domain. Let $\tilde{f} \in C^0(\mathbb{R})$ be a continuous function satisfying*

$$\begin{cases} \exists\, \mu \geq 0 \quad : \quad -\tilde{f}(t) \geq h(t) \quad \text{for} \quad t > \mu, \\ \\ \text{where } h \in C^0([0,+\infty)) \text{ satisfies the Keller-Osserman condition (2.5.2).} \end{cases} \tag{2.5.4}$$

Let $v \in C^0(\overline{\Omega}) \cap H^1_{loc}(\overline{\Omega})$ be a solution to

$$\begin{cases} -\Delta v \leq \tilde{f}(v) & \text{in} \quad \mathcal{D}'(\Omega), \\ v \leq \mu & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5.5}$$

then, $v \leq \mu$ in Ω.

In particular, any $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5.6}$$

where f is a specified Allen-Cahn type nonlinearity, satisfies $u \leq 1$ in Ω.

Proof. By assumption (2.5.4) and Kato's inequality, we have

$$\Delta(v-\mu)^+ \geq -\tilde{f}(v)\mathbf{1}_{\{v>\mu\}} \geq h(v)\mathbf{1}_{\{v>\mu\}} \geq h((v-\mu)^+) \quad \text{in} \quad \mathcal{D}'(\Omega)$$

and $(v-\mu)^+ = 0$ on $\partial\Omega$. Then, $(v-\mu)^+ = 0$ on Ω by Proposition 2.5.2.

The second part of the claim follows from the first one, by taking $\tilde{f}(t) = f(t+1)$, $h(t) = ct^p$, $\mu = 0$ and $v = u-1$. Indeed, the assumption `W4`) implies that (2.5.4) is satisfied (recall that $p > 1$ and $c > 0$ in `W4`)). □

Theorem 2.5.1. *Let $\Omega \subset \mathbb{R}^N$ be a proper open set of class C^1 and η its outward unit normal. Assume $f \in L^\infty(\Omega)$ and let $v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to*

$$\begin{cases} -\Delta v = f & \text{in} \quad \Omega, \\ v \geq 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then, there exists a positive constant C, depending only on N, such that

$$\forall\, x \in \Omega \qquad |\nabla v(x)| \leq C\Big(\Big|\frac{\partial v}{\partial \eta}(x')\Big| + d(x,\partial\Omega)\|f\|_{L^\infty(B(x,d_x))}\Big),$$

where $x' \in \partial\Omega$ is such that $|x - x'| = d(x,\partial\Omega) =: d_x$.

Proof. Let $x \in \Omega$ and $x' \in \partial\Omega$ such that $|x - x'| = d(x, \partial\Omega)$.
According to Theorem 8.34 in [GT01], there exists $u \in C^{1,\alpha}(\overline{B(x, d_x)})$ a weak solution to

$$\begin{cases} -\Delta u = f & \text{in} \quad B(x, d_x), \\ u = 0 & \text{on} \quad \partial B(x, d_x). \end{cases}$$

By Lemma 2.5.3, there is $C_1 = C_1(N) > 0$ such that

$$\|\nabla u\|_{L^\infty(B(x,d_x))} \leq C_1 d_x \|f\|_{L^\infty(B(x,d_x))}. \tag{2.5.7}$$

Now we set $h = v - u$, then $h \in C^2(B(x, d_x)) \cap C^1(\overline{B(x, d_x)})$ and

$$\begin{cases} -\Delta h = 0 & \text{in} \quad B(x, d_x), \\ h \geq 0 & \text{on} \quad \overline{B(x, d_x)}, \end{cases}$$

where the latter follows by the maximum principle, since $h = v \geq 0$ on $\partial B(x, d_x) \subset \overline{\Omega}$.

Therefore, by Lemma 2.5.4, there exists a positive constant $C_2 = C_2(N) > 0$ such that

$$|\nabla h(x)| \leq C_2 \Big| \frac{\partial h}{\partial \eta}(x') \Big|. \tag{2.5.8}$$

Finally, by (2.5.7) and (2.5.8), we obtain

$$\begin{aligned} |\nabla v(x)| &\leq |\nabla u(x)| + |\nabla h(x)| \leq C_1 d_x \|f\|_{L^\infty(B(x,d_x))} + C_2 \Big| \frac{\partial h}{\partial \eta}(x') \Big| \\ &\leq C_1 d_x \|f\|_{L^\infty(B(x,d_x))} + C_2 \Big| \frac{\partial u}{\partial \eta}(x') \Big| + C_2 \Big| \frac{\partial v}{\partial \eta}(x') \Big| \\ &\leq (1 + C_2) C_1 d_x \|f\|_{L^\infty(B(x,d_x))} + C_2 \Big| \frac{\partial v}{\partial \eta}(x') \Big| \\ &\leq \max\{C_2, (1 + C_2)C_1\} \Big(d_x \|f\|_{L^\infty(B(x,d_x))} + \Big| \frac{\partial v}{\partial \eta}(x') \Big| \Big). \end{aligned}$$

□

Corollary 2.5.2. *Let $\Omega \subset \mathbb{R}^N$ be a proper open set of class C^1. Let $f \in C^0(\mathbb{R})$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a bounded solution to* (2.1.1).
Then $\nabla u \in L^\infty(\Omega)$ and

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(|\mathfrak{c}| + \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty([0, \|u\|_{L^\infty(\Omega)}])}),$$

where $C > 0$ is a constant depending only on N.

Proof. Let $x \in \Omega$.

1) If $d(x, \partial\Omega) \geq 1$, then $B(x, 1) \subset \Omega$. Hence, we can apply a standard interior gradient estimate (see for instance Theorem 3.9 in [GT01]) to get

$$|\nabla u(x)| \leq C_1(N)(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty([0, \|u\|_{L^\infty(\Omega)}])}). \tag{2.5.9}$$

2) If $d(x, \partial\Omega) < 1$, let $x' \in \partial\Omega$ be such that $d(x, \partial\Omega) = |x - x'|$. Then, Theorem 2.5.1 implies that

$$|\nabla u(x)| \leq C_2(N)(|\mathfrak{c}| + \|f\|_{L^\infty([0, \|u\|_{L^\infty(\Omega)}])}). \tag{2.5.10}$$

The desired conclusion then follows from (2.5.9) and (2.5.10). □

2.5.2 Some uniform estimates

Assume $N \geq 2$. In the following, for a continuous function $h : \mathbb{R}^{N-1} \to \mathbb{R}$, we consider its epigraph

$$\omega := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \, : \, x_N > h(x') \right\},$$

and, for any $R > 0$ and any $\tau > \sup\limits_{B'(0',R)} |h|$, we set

$$\mathfrak{C}^h(0', R, \tau) = \left\{ x = (x', x_N) \in \mathbb{R}^N, \, x' \in B'(0', R) \text{ and } h(x') < x_N < \tau \right\}, \tag{2.5.11}$$

the intersection of the epigraph ω with the truncated cylinder $B'(0', R) \times (-\tau, \tau)$ and

$$\widehat{\mathfrak{C}^h}(0', R, \tau) = \mathfrak{C}^h(0', R, \tau) \cup \{ x = (x', x_N) \in B'(0', R) \times \mathbb{R}, \, x_N = h(x') \} \, .$$

Then, the next uniform estime holds.

Theorem 2.5.2. *Assume $\sigma \in (0, 1]$, $c \in \mathbb{R}$, $\mathcal{E} > 0$ and let $h \in \mathbb{C}^{1,\sigma}(\mathbb{R}^{N-1})$ be such that $\|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})} \leq \mathcal{E}$.*
Let $\widetilde{R} > 0$, $\widetilde{\tau} > \max \left\{ \widetilde{R}, \, 4 \underline{\sup_{B'(0',\widetilde{R})}} |h| \right\}$ and $u \in C^1(\overline{\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})}) \cap H^2(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau}))$ satisfy

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathcal{D}'(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})), \\ \quad u = 0 & \text{on} \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0', \widetilde{R}, \widetilde{\tau}), \\ \dfrac{\partial u}{\partial \eta} = c & \text{on} \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0', \widetilde{R}, \widetilde{\tau}), \end{cases} \tag{2.5.12}$$

where $f \in Lip_{loc}(\mathbb{R})$.
Then, there exists $\alpha = \alpha(N, \mathcal{E}, \sigma) \in (0, 1)$ such that $\nabla u \in C^{0,\alpha}(\overline{\mathfrak{C}^h(0', r, t)})$ for any $0 < r < \frac{\widetilde{R}}{2}$ and any $\sup\limits_{B'(0',r)} |h| < t < \dfrac{3}{4} \widetilde{\tau}$,

and

$$[\nabla u]_{C^{0,\alpha}(\overline{\mathfrak{C}^h(0',r,t)})} \leq C \left[|c| \left(1 + \|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})} \right)^2 + 1 + \|f'\|_{L^\infty([-M,M])} \right] (1 + \|\nabla u\|_{L^\infty(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau}))}), \tag{2.5.13}$$

where $C = C(r, \widetilde{R}, \widetilde{\tau}, \alpha, N) > 0$ and $M = \|u\|_{L^\infty(\mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau}))}$.

The proof of the above theorem is based on the following general result.

Proposition 2.5.3. *Let ω be the epigraph of a continuous function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ and suppose that ω satisfies a uniform exterior cone condition (with reference cone V).*[7]

[7] Let us recall that an open set $\omega \subset \mathbb{R}^N$ (not necessarily an epigraph) satisfies a *uniform exterior cone condition* if for any $x_0 \in \partial\omega$ there exists a finite right circular cone V_{x_0}, with vertex x_0, such that $\overline{\omega} \cap V_{x_0} = \{x_0\}$ and the cones V_{x_0} are all congruent to some fixed cone V. The cone V is called the *reference cone*.

Let $\widetilde{R} > 0$, $\widetilde{\tau} > \max\left\{\widetilde{R}, \sup_{\overline{B'(0',\widetilde{R})}} |h|\right\}$, $f \in L^N(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))$, $\Psi \in C^{0,\beta}(\partial\omega \cap \widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))$ *and* $v \in H^1(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})) \cap C^0(\overline{\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})})$ *satisfy*

$$\begin{cases} -\Delta v = f & \text{in} \quad \mathcal{D}'(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})), \\ v = \Psi & \text{on} \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}), \end{cases} \tag{2.5.14}$$

Then, there are constants $C = C(N, V, \widetilde{R}, \beta) > 0$ *and* $\alpha = \alpha(N, V, \beta) \in (0,1)$ *such that for any* $r > 0$ *and for any* $x_0 \in \partial\omega \cap \left(\overline{B'(0',\widetilde{R}/2)} \times \mathbb{R}\right)$,

$$\underset{\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau})\cap B(x_0,r)}{osc} v \leq C([\Psi]_{C^{0,\beta}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))} + \|v\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))} + \|f\|_{L^N(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))})r^\alpha. \tag{2.5.15}$$

Proof. Let $x, y \in \widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}) \cap B(x_0, r)$.

If $r \geq \min\left\{1, \frac{\widetilde{R}}{2}\right\} =: R_0 > 0$, then

$$|v(x) - v(y)| \leq 2\|v\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))} \leq 2R_0^{-\gamma}\|v\|_{L^\infty(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau}))} r^\gamma, \tag{2.5.16}$$

for any $\gamma \in (0,1)$.

If $r < R_0$, we apply Theorem 8.27 of [GT01] with $\Omega = \mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})$, $q = 2N$ and $g = f$. Hence, there are positive constants $C' = C'(N, V)$ and $\theta = \theta(N, V)$ such that

$$\underset{\Omega\cap B(x_0,r)}{\text{osc}} v \leq C'\left\{r^\theta(R_0^{-\theta}\|v\|_{L^\infty(\Omega)} + \|f\|_{L^N(\Omega)}) + \sigma(\sqrt{rR_0})\right\},$$

where $\sigma(\sqrt{rR_0}) = \underset{\partial\Omega\cap B(x_0,\sqrt{rR_0})}{\text{osc}} v = \underset{\partial\omega\cap B(x_0,\sqrt{rR_0})}{\text{osc}} \Psi \leq 2^\beta\,[\Psi]_{C^{0,\beta}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))}\, R_0^{\beta/2} r^{\beta/2}$.
Therefore,

$$\underset{\Omega\cap B(x_0,r)}{\text{osc}} v \leq C'(2^\beta + R_0^{-\theta})([\Psi]_{C^{0,\beta}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))} + \|v\|_{L^\infty(\Omega)} + \|f\|_{L^N(\Omega)})r^\alpha, \tag{2.5.17}$$

where $\alpha = \min(\theta, \frac{\beta}{2})$.
The desired inequality (2.5.15) then follows from (2.5.16) with $\gamma = \alpha$ and (2.5.17). □

Now we are ready to prove Theorem 2.5.2.

Proof of Theorem 2.5.2. First, we observe that $w_i := \frac{\partial u}{\partial x_i} \in H^1(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})) \cap C^0(\overline{\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})})$ solves

$$\begin{cases} -\Delta w_i = f'(u)w_i & \text{in} \quad \mathcal{D}'(\mathfrak{C}^h(0',\widetilde{R},\widetilde{\tau})), \\ w_i = c\eta_i & \text{on} \quad \partial\omega \cap \widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}), \end{cases}$$

where η is the outward unit normal at $\partial\omega$. Since

$$\Psi_i(x') := \eta_i(x', h(x')) = \begin{cases} \frac{\partial_i h(x')}{\sqrt{1+|\nabla h(x')|^2}} & \text{if} \quad i \in \{1, \cdots, N-1\}, \\ \frac{-1}{\sqrt{1+|\nabla h(x')|^2}} & \text{if} \qquad\quad i = N, \end{cases}$$

for any $x' \in \mathbb{R}^{N-1}$, it is easily seen that $\Psi_i \in C^{0,\sigma}(B'(0', \widetilde{R}))$ and

$$[\Psi_i]_{C^{0,\sigma}(B'(0',\widetilde{R}))} \leq \left(1 + \|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})}\right) [\nabla h]_{C^{0,\sigma}(\mathbb{R}^{N-1})}\,.$$

Hence,

$$\left[w_{i_{|\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau})}}\right]_{C^{0,\sigma}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))} \leq |c| \left(1 + \|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})}\right)^2. \tag{2.5.18}$$

Moreover, since h is globally Lipschitz-continuous on $\mathbb{R}^{N-1}$, the epigraph ω satisfies a uniform exterior cone condition with reference cone V depending only on $\|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})}$, hence, only on $\mathcal{E}$.[8]

Set $\Omega = \mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})$, $R_1 = \min(\frac{\widetilde{R}}{2} - r, \frac{\widetilde{\tau}}{2})$, $M = \|u\|_{L^\infty(\Omega)}$ and pick $x, y \in \mathfrak{C}^h(0', r, t)$ with $x \neq y$.

1. If $|x-y| \geq \dfrac{R_1}{4}$ then, for any $\gamma \in (0,1)$, we have

$$\begin{aligned} |w_i(x) - w_i(y)| &\leq 2\|\nabla u\|_{L^\infty(\Omega)} = 2\|\nabla u\|_{L^\infty(\Omega)} \Big(\frac{4}{R_1}\Big)^\gamma \Big(\frac{R_1}{4}\Big)^\gamma \\ &\leq 2\Big(\frac{4}{R_1}\Big)^\gamma \|\nabla u\|_{L^\infty(\Omega)}|x-y|^\gamma := M_1\|\nabla u\|_{L^\infty(\Omega)}|x-y|^\gamma. \end{aligned} \tag{2.5.19}$$

2. If $|x-y| < \dfrac{R_1}{4}$, then we consider the set

$$T := \left\{x = (x', x_N) \in \mathbb{R}^N \,:\, x' \in \overline{B'(0', \widetilde{R}/2)} \quad \text{and} \quad x_N = h(x')\right\}$$

and, by observing that x and y play a symmetric role, we distinguish (only) three cases.

Case 2.1 : $d(y,T) > \frac{R_1}{2}$.

In this case we have $\overline{B(y, \frac{R_1}{2})} \subset \mathfrak{C}^h(0', \widetilde{R}, \widetilde{\tau})$. To see this, we first observe that $\overline{B(y, \frac{R_1}{2})} \subset \omega$, thanks to the assumption $d(y,T) > \frac{R_1}{2}$ and by the definition of R_1, and thus, for any $z \in \overline{B(y, \frac{R_1}{2})}$, we have

$$h(z') < z_N = z_N - y_N + y_N < z_N - y_N + t \leq \frac{R_1}{2} + \frac{3\widetilde{\tau}}{4} < \frac{\widetilde{\tau}}{4} + \frac{3\widetilde{\tau}}{4} = \widetilde{\tau},$$

[8] more precisely, we can choose the same reference cone $V = V(\mathcal{E})$, for any globally Lipschitz-continuous function h such that $\|\nabla h\|_{L^\infty(\mathbb{R}^{N-1})} \leq \mathcal{E}$.

and

$$|z'| \leq |z' - y'| + |y'| \leq \frac{R_1}{2} + r < \frac{\widetilde{R}}{2} - r + r < \widetilde{R}.$$

Now, by applying Lemma 2.5.5 with $\delta = \frac{R_1}{2}$ and $\gamma \in (0,1)$, we deduce

$$\begin{aligned} |w_i(x) - w_i(y)| &\leq \frac{\sqrt{N}}{2^{1-\gamma}}\Big(2N\|\nabla u\|_{L^\infty(\Omega)}\Big(\frac{R_1}{2}\Big)^{-\gamma} + \|f'\|_{L^\infty([-M,M])}\|\nabla u\|_{L^\infty(\Omega)}\Big(\frac{R_1}{2}\Big)^{2-\gamma}\Big)|x-y|^\gamma \\ &\leq M_2(1 + \|f'\|_{L^\infty([-M,M])})\|\nabla u\|_{L^\infty(\Omega)}|x-y|^\gamma, \end{aligned} \tag{2.5.20}$$

where $M_2 = \dfrac{\sqrt{N}}{2^{1-\gamma}}\Big(2N\Big(\dfrac{R_1}{2}\Big)^{-\gamma} + \Big(\dfrac{R_1}{2}\Big)^{2-\gamma}\Big)$.

Case 2.2 : $d(x,T) \leq d(y,T) \leq \dfrac{R_1}{2}$ and $|x-y| \geq \frac{1}{4}d(y,T)$.

Since T is a compact set, there exists $y_0 = (y_0', h(y_0')) \in T$ such that $d(y,T) = |y - y_0|$. Therefore

$$x, y \in B(y_0, 6|x-y|),$$

since

$$|x - y_0| \leq |x-y| + |y-y_0| \leq |x-y| + 4|x-y| = 5|x-y|,$$

and

$$|y - y_0| = d(y,T) \leq 4|x-y|.$$

Now, we apply Proposition 2.5.3 with $r = 6|x-y|$, $v = w_i$, $\Psi = c\eta_i$ and $x_0 = y_0$, thus there exist constants $C = C(N, \mathcal{E}, \widetilde{R}, \sigma) > 0$ and $\alpha = \alpha(N, \mathcal{E}, \sigma) \in (0,1)$ such that

$$\begin{aligned} |w_i(x) - w_i(y)| &\leq \operatorname*{osc}_{\Omega \cap B(x_0,r)} w_i \\ &\leq 6^\alpha C\left(\Big[w_{i_{|\partial\omega \cap \widehat{\mathfrak{C}}^h(0',\widetilde{R},\widetilde{\tau})}}\Big]_{C^{0,\sigma}(\partial\omega \cap \widehat{\mathfrak{C}}^h(0',\widetilde{R},\widetilde{\tau}))} + \|w_i\|_{L^\infty(\Omega)} + \|f'(u)w_i\|_{L^N(\Omega)}\right)|x-y|^\alpha \\ &\leq 6^\alpha C\left(|c|\left(1 + \|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})}\right)^2 + \|\nabla u\|_{L^\infty(\Omega)} + \|f'(u)w_i\|_{L^N(\Omega)}\right)|x-y|^\alpha, \end{aligned} \tag{2.5.21}$$

where in the latter we have used (2.5.18).

Since

$$\|f'(u)w_i\|_{L^N(\Omega)} \leq \|f'\|_{L^\infty([-M,M])}\|\nabla u\|_{L^\infty(\Omega)}\left(2\widetilde{\tau}\mathcal{L}^{N-1}(B'(0',1))\widetilde{R}^{N-1}\right)^{\frac{1}{N}},$$

where $\mathcal{L}^{N-1}(B'(0',1))$ denotes the measure of the unit ball of $\mathbb{R}^{N-1}$, we see that

$$|w_i(x) - w_i(y)| \leq M_3M_3'(1 + \|\nabla u\|_{L^\infty(\Omega)})|x-y|^\alpha, \tag{2.5.22}$$

where

$$\begin{cases} M_3 = 6^\alpha C\left[1+\left(2\widetilde{\tau}\mathcal{L}^{N-1}(B'(0',1))\widetilde{R}^{N-1}\right)^{\frac{1}{N}}\right], \\ M_3' = \left[|c|\left(1+\|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})}\right)^2+1+\|f'\|_{L^\infty([-M,M])}\right]. \end{cases} \tag{2.5.23}$$

Case 2.3: $d(x,T)\le d(y,T)\le \dfrac{R_1}{2}$ and $|x-y|<\frac{1}{4}d(y,T)$.

Let $y_0\in T$ such that $|y-y_0|=d(y,T)$. The function $v_i:=w_i-w_i(y_0)$ satisfies

$$-\Delta v_i = f'(u)w_i.$$

By applying Lemma 2.5.5 with $\delta=\frac{d(y,T)}{2}$ and $\gamma=\alpha$, we have

$$\begin{aligned} |w_i(x)-w_i(y)| = |v_i(x)-v_i(y)| \le \frac{\sqrt{N}}{2^{1-\alpha}}\Bigg(&\|f'\|_{L^\infty([-M,M])}\|\nabla u\|_{L^\infty(\Omega)}\left(\frac{d(y,T)}{2}\right)^{2-\alpha} \\ &+2N\|v_i\|_{L^\infty(\overline{B(y,\frac{d(y,T)}{2})})}\left(\frac{d(y,T)}{2}\right)^{-\alpha}\Bigg)|x-y|^\alpha. \end{aligned} \tag{2.5.24}$$

Now, we want to estimate $\|v_i\|_{L^\infty(\overline{B(y,\frac{d(y,T)}{2})})}$. Let $z\in\overline{B(y,\frac{d(y,T)}{2})}$ and $y_0=(y_0',h(y_0'))\in T$ such that $d(y,T)=|y_0-y|$. Then $z\in\overline{\Omega}\cap B(y_0,2d(y,T))$ since

$$|z-y_0| = |z-y+y-y_0| \le |z-y|+|y-y_0| \le \frac{3}{2}d(y,T) < 2d(y,T).$$

Thus we can apply Proposition 2.5.3 with $r=2d(y,T)$, $v=v_i$, $\Psi=c(\eta_i-\eta_i(y_0))$ and $x_0=y_0$ to get

$$\begin{aligned} &|v_i(z)| = |v_i(z)-v_i(y_0)| \\ &\le 2^\alpha C\left(\left[v_{i_{|\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau})}}\right]_{C^{0,\sigma}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))}+\|v_i\|_{L^\infty(\Omega)}+\|f'(u)w_i\|_{L^N(\Omega)}\right)d(y,T)^\alpha \\ &\le 2^\alpha C\left(\left[w_{i_{|\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau})}}\right]_{C^{0,\sigma}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))}+2\|w_i\|_{L^\infty(\Omega)}+\|f'(u)w_i\|_{L^N(\Omega)}\right)d(y,T)^\alpha \\ &\le 2^{\alpha+1} C\left(\left[w_{i_{|\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau})}}\right]_{C^{0,\sigma}(\partial\omega\cap\widehat{\mathfrak{C}^h}(0',\widetilde{R},\widetilde{\tau}))}+\|w_i\|_{L^\infty(\Omega)}+\|f'(u)w_i\|_{L^N(\Omega)}\right)d(y,T)^\alpha \\ &\le 3^{-\alpha}2M_3\left[|c|\left(1+\|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})}\right)^2+1+\|f'\|_{L^\infty([-M,M])}\right](1+\|\nabla u\|_{L^\infty(\Omega)})d(y,T)^\alpha \\ &\le 3^{-\alpha}2M_3M_3'(1+\|\nabla u\|_{L^\infty(\Omega)})d(y,T)^\alpha. \end{aligned}$$

The latter and (2.5.24) yield

$$
\begin{aligned}
|w_i(x) - w_i(y)| \le \frac{\sqrt{N}}{2^{1-\alpha}} \Big(\|f'\|_{L^\infty([-M,M])} \|\nabla u\|_{L^\infty(\Omega)} \Big(\frac{d(y,T)}{2}\Big)^{2-\alpha} \\
+ 4N3^{-\alpha} M_3 M_3' (1 + \|\nabla u\|_{L^\infty(\Omega)}) d(y,T)^\alpha \Big(\frac{d(y,T)}{2}\Big)^{-\alpha} \Big) |x-y|^\alpha \\
\le \frac{\sqrt{N}}{2^{1-\alpha}} \Big(\|f'\|_{L^\infty([-M,M])} \|\nabla u\|_{L^\infty(\Omega)} \Big(\frac{R_1}{4}\Big)^{2-\alpha} + 4N M_3 M_3' (1 + \|\nabla u\|_{L^\infty(\Omega)}) \Big) |x-y|^\alpha \\
\le \frac{\sqrt{N}}{2^{1-\alpha}} \Big(M_3' \Big(\frac{R_1}{4}\Big)^{2-\alpha} + 4N M_3 M_3' \Big) (1 + \|\nabla u\|_{L^\infty(\Omega)}) |x-y|^\alpha \\
\le M_4 \left[|c| \left(1 + \|\nabla h\|_{C^{0,\sigma}(\mathbb{R}^{N-1})}\right)^2 + 1 + \|f'\|_{L^\infty([-M,M])} \right] (1 + \|\nabla u\|_{L^\infty(\Omega)}) |x-y|^\alpha,
\end{aligned}
\tag{2.5.25}
$$

where $M_4 = \frac{\sqrt{N}}{2^{1-\alpha}} \Big(\Big(\frac{R_1}{4}\Big)^{2-\alpha} + 4N M_3 \Big)$.

The desired conclusion (2.5.13) then follows from (2.5.19), (2.5.20) with $\gamma = \alpha$, and (2.5.22), (2.5.25) by taking $C = \max(M_1, M_2, M_3, M_4)$. ☐

2.5.3 Some energy estimates

In this subsection we employ the strategy developed in [FV10] (see Section 9 therein). To this end we recall that have denoted by F the primitive of f vanishing at 0.

Lemma 2.5.1. *Assume $c \in \mathbb{R}$ and $N \ge 2$. Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a bounded solution to* (2.1.1) *where $\Omega \subset \mathbb{R}^N$ is an epigraph with boundary of class C^1 and $f \in Lip_{loc}([0,+\infty))$.*

If u is monotone, i.e.,

$$
\frac{\partial u}{\partial x_N}(x) > 0 \qquad \forall x \in \Omega, \tag{2.5.26}
$$

then there exists a constant $C > 0$, depending only on $\mathfrak{c}$, N, f and $\|u\|_{L^\infty(\Omega)}$, such that, for any $R > 0$,

$$
\int_{B(0,R)\cap\Omega} \frac{|\nabla u|^2}{2} + (c - F(u)) \le C \left(\mathcal{H}^{N-1}(\partial(\Omega \cap B(0,R))) + \int_{B(0,R)\cap\Omega} \frac{|\nabla \overline{u}|^2}{2} + (c - F(\overline{u})) \right), \tag{2.5.27}
$$

where[9] $\overline{u}(x', x_N) = \overline{u}(x') =: \lim_{x_N \to +\infty} u(x', x_N)$.

Proof. From Corollary 2.5.2 there exists a constant $M > 0$, depending only on $\mathfrak{c}$, N, f and $\|u\|_{L^\infty(\Omega)}$, such that

$$
u(x) + |\nabla u(x)| \le M \quad \text{for any} \quad x \in \overline{\Omega}. \tag{2.5.28}
$$

[9] $\overline{u}$ is well-defined on $\mathbb{R}^N$ since u is bounded and monotone on Ω. Also recall that, by standard elliptic estimates, $\overline{u}$ solves the equation $-\Delta \overline{u} = f(\overline{u})$ on $\mathbb{R}^N$, and thus also on $\mathbb{R}^{N-1}$.

As in [FV10], for any $R > 0$ we consider the energy

$$\mathcal{E}_{R,c}(u) := \int_{B(0,R)\cap\Omega} \frac{|\nabla u|^2}{2} + (c - F(u))$$

and, for any $x = (x', x_N) \in \overline{\Omega}$ and $t \geq 0$, we define

$$u^t(x', x_N) := u(x', x_N + t).$$

We can therefore proceed as in Lemma 9.1. of [FV10] to get, for any $R, T > 0$,

$$\mathcal{E}_{R,c}(u^T) - \mathcal{E}_{R,c}(u) \geq -2M^2 \int_{\partial(\Omega\cap B(0,R))} d\mathcal{H}^{N-1} = -2M^2\mathcal{H}^{N-1}(\partial(\Omega \cap B(0,R))).$$

The desired conclusion then follows by letting $T \longrightarrow \infty$ in the latter inequality, for any fixed $R > 0$. $\square$

When $N = 3$ and f is of class C^1, the above result can be better specified as follows :

Lemma 2.5.2. *Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a bounded solution to (2.1.1) where $\Omega \subset \mathbb{R}^3$ is an epigraph with boundary of class C^1. Let $f \in C^1([0,+\infty))$ be such that*

$$F(\sup u) = \sup_{t\in[0,\sup u]} F(t). \tag{2.5.29}$$

If u is monotone, i.e.,

$$\frac{\partial u}{\partial x_3}(x) > 0 \qquad \forall x \in \Omega, \tag{2.5.30}$$

then there exists a constant $C > 0$, depending only on $\mathfrak{c}$, f and $\|u\|_{L^\infty(\Omega)}$, such that

$$\int_{B(0,R)\cap\Omega} |\nabla u|^2 \leq C\left(R^2 + \mathcal{H}^2(\partial(\Omega \cap B(0,R))\right) \qquad \forall\, R > 0. \tag{2.5.31}$$

Proof. Set $c_u = \sup_{t\in[0,\sup u]} F(t)$. Then, from Lemma 2.5.1, applied with $c = c_u$ and $N = 3$, we have for any $R > 0$,

$$\int_{B(0,R)\cap\Omega} \frac{|\nabla u|^2}{2} + (c - F(u)) \leq C\left(\mathcal{H}^2(\partial(\Omega\cap B(0,R))) + \int_{B(0,R)\cap\Omega} \frac{|\nabla \overline{u}|^2}{2} + (c - F(\overline{u}))\right). \tag{2.5.32}$$

Now we estimate $\int_{B(0,R)\cap\Omega} \frac{|\nabla \overline{u}|^2}{2} + (c - F(\overline{u}))$, for any $R > 0$. To this end, we first observe that the monotonicity assumption (2.5.30) implies that u is a stable solution to $-\Delta u = f(u)$ in Ω. Hence, since f is of class C^1, the limit profil $\overline{u}$ is a classical *stable solution* to

$$\begin{cases} -\Delta \overline{u} = f(\overline{u}) & \text{in } \mathbb{R}^2, \\ 0 < \overline{u} \leq \sup u & \text{in } \mathbb{R}^2, \\ \sup_{\mathbb{R}^2} \overline{u} = \sup_\Omega u. \end{cases} \tag{2.5.33}$$

It is well-known (cf. [Dan05], see also [FSV08] for related results) that any bounded entire two-dimensional stable solution to $-\Delta v = f(v)$, with $f \in C^1$, is either constant or one-dimensional and strictly monotone with respect to some fixed direction.

If $\overline{u}$ is constant, we have $\overline{u} \equiv \sup_{\mathbb{R}^2} \overline{u} = \sup_\Omega u$ and so $F(\overline{u}) = F(\sup u) = c_u$, by assumption. Hence, $\frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) \equiv 0$ and $\int_{B(0,R)\cap\Omega} \frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) = 0$ for any $R > 0$. Then, (2.5.31) follows immediately from (2.5.32).

If $\overline{u}$ is not constant, then after a rotation of the coordinates, we have

$$\overline{u}(x_1, x_2) = \overline{u}(x_1) \quad \text{for any } (x_1, x_2) \in \mathbb{R}^2 \tag{2.5.34}$$

and $\overline{u}$ is a strictly monotone solution to the ODE

$$-\overline{u}'' = f(\overline{u}) \quad \text{in} \quad \mathbb{R}. \tag{2.5.35}$$

Therefore,

$$\frac{(\overline{u}'(s))^2}{2} + F(\overline{u}(s)) = F(\sup \overline{u}) \qquad \forall s \in \mathbb{R}, \tag{2.5.36}$$

and so, by (2.5.33), we also have

$$\frac{(\overline{u}'(s))^2}{2} + F(\overline{u}(s)) = F(\sup \overline{u}) = c_u \qquad \forall s \in \mathbb{R}. \tag{2.5.37}$$

From the latter we infer that

$$c_u - F(\overline{u}(s)) > 0 \qquad \forall s \in \mathbb{R}, \tag{2.5.38}$$

and

$$\frac{(\overline{u}'(s))^2}{2} = c_u - F(\overline{u}(s)) \leq F(\sup u) - \inf_{t\in[0,\sup u]} F(t) = C(f, \sup u) \qquad \forall s \in \mathbb{R}, \tag{2.5.39}$$

hence,

$$\begin{aligned}
&\int_{B(0,R)\cap\Omega} \frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) \leq \int_{B(0,R)} \frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) \\
&\leq \int_{(-R,R)^3} \frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) = R^2 \int_{-R}^{R} \left[\frac{(\overline{u}'(x_1))^2}{2} + (c_u - F(\overline{u}(x_1))) \right] dx_1.
\end{aligned}$$

From (2.5.37) we deduce that

$$\int_{B(0,R)} \frac{|\nabla \overline{u}|^2}{2} + (c_u - F(\overline{u})) \leq R^2 \int_{-R}^{R} (\overline{u}'(x_1))^2 dx_1 \leq R^2 \|\overline{u}'\|_{L^\infty(\mathbb{R})} \int_{-R}^{R} |\overline{u}'(x_1)| dx_1. \tag{2.5.40}$$

If $\overline{u}' > 0$, then

$$\int_{-R}^{R} |\overline{u}'(x_1)| dx_1 = \int_{-R}^{R} \overline{u}'(x_1) dx_1 = \overline{u}(R) - \overline{u}(-R) \leq 2\|\overline{u}\|_{L^\infty(\mathbb{R})}, \tag{2.5.41}$$

if $\overline{u}' < 0$, then

$$\int_{-R}^{R} |\overline{u}'(x_1)|dx_1 = \int_{R}^{-R} \overline{u}'(x_1)dx_1 = \overline{u}(-R) - \overline{u}(R) \leq 2\|\overline{u}\|_{L^\infty(\mathbb{R})}. \tag{2.5.42}$$

In both cases we have

$$\int_{B(0,R)} \frac{|\nabla\overline{u}|^2}{2} + (c_u - F(\overline{u})) \leq 2\|\overline{u}'\|_{L^\infty(\mathbb{R})}\|\overline{u}\|_{L^\infty(\mathbb{R})}R^2 \tag{2.5.43}$$

and so, from (2.5.32), (2.5.29) and (2.5.43), we deduce that

$$\begin{aligned}
&\int_{B(0,R)\cap\Omega} \frac{|\nabla u|^2}{2} \leq \int_{B(0,R)\cap\Omega} \frac{|\nabla u|^2}{2} + (c_u - F(u)) \\
&\leq C\left(\mathcal{H}^2(\partial(\Omega\cap B(0,R))) + \int_{B(0,R)} \frac{|\nabla\overline{u}|^2}{2} + (c_u - F(\overline{u}))\right) \\
&\leq C\left(\mathcal{H}^2(\partial(\Omega\cap B(0,R))) + 2\|\overline{u}'\|_{L^\infty(\mathbb{R})}\|\overline{u}\|_{L^\infty(\mathbb{R})}R^2\right),
\end{aligned}$$

which, in view of (2.5.39), concludes the proof. □

2.5.4 Appendix

Lemma 2.5.3. *Assume $\alpha \in (0,1)$, $x_0 \in \mathbb{R}^N$, $R > 0$ and $f \in L^\infty(B(x_0,R))$. Let $u \in C^{1,\alpha}(\overline{B(x_0,R)})$ be a weak solution to*

$$\begin{cases} -\Delta u = f & \text{in} \quad B(x_0,R), \\ u = 0 & \text{on} \quad \partial B(x_0,R). \end{cases}$$

Then, there exists a positive constant C, depending only on N, such that

$$\|\nabla u\|_{L^\infty(B(x_0,R))} \leq C\|f\|_{L^\infty(B(x_0,R))}R.$$

Proof. The function

$$w(x) = u(x_0 + Rx), \qquad x \in B_1 := B(0,1),$$

is a weak solution to

$$\begin{cases} -\Delta w = g & \text{in} \quad B_1, \\ w = 0 & \text{on} \quad \partial B_1, \end{cases}$$

where $g(x) = R^2 f(x_0 + Rx)$, $x \in B_1$.

Therefore, according to Theorem 8.33 in [GT01], there is a constant $\widetilde{C} > 0$, depending only on N, such that

$$\|\nabla w\|_{L^\infty(B_1)} \leq \widetilde{C}(\|w\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)}). \tag{2.5.44}$$

Now, the function

$$v(x) = w(x) + \frac{\|g\|_{L^\infty(B_1)}}{2N}|x|^2, \qquad x \in B_1$$

satisfies $v \geq w$ on B_1 and

$$-\Delta v = -\Delta w - \|g\|_{L^\infty(B_1)} = g - \|g\|_{L^\infty(B_1)} \leq 0.$$

Hence, the weak maximum principle (see for instance Theorem 8.1 in [**gt**]) yields

$$w(x) \leq \sup_{\partial B_1} v = \frac{\|g\|_{L^\infty(B_1)}}{2N}, \qquad x \in B_1, \tag{2.5.45}$$

and so,

$$\|\nabla w\|_{L^\infty(B_1)} \leq 2\widetilde{C}\|g\|_{L^\infty(B_1)} \leq 2\widetilde{C}R^2\|f\|_{L^\infty(B(x_0,R))}. \tag{2.5.46}$$

Then, by definition of w we get

$$\|\nabla u\|_{L^\infty(B(x_0,R))} = \frac{\|\nabla w\|_{L^\infty(B_1)}}{R} \leq 2\widetilde{C}\|f\|_{L^\infty(B(x_0,R))}R.$$

□

Lemma 2.5.4. *Let $x_0 \in \mathbb{R}^N$, $R > 0$ and $h \in C^2(B(x_0,R)) \cap C^1(\overline{B(x_0,R)})$ be a solution to*

$$\begin{cases} -\Delta h = 0 & \text{in} \quad B(x_0,R), \\ h \geq 0 & \text{in} \quad \overline{B(x_0,R)}. \end{cases} \tag{2.5.47}$$

If $x \in \partial B(x_0,R)$ and $h(x) = 0$, then there is a positive constant C, depending only on N, such that

$$|\nabla h(x_0)| \leq C\left|\frac{\partial h}{\partial \eta}(x)\right|,$$

where η is the outward unit normal at $\partial B(x_0,R)$.

Proof. As in the previous Lemma we can suppose that $x_0 = 0$ and $R = 1$. Also, we can suppose that $h(0) > 0$, otherwise the result is trivially true by the strong maximum principle.
By Harnack inequality, there is a constant $C_H = C_H(N) > 0$ such that

$$h(0) \leq \sup_{\overline{B(0,1/2)}} h \leq C_H \inf_{\overline{B(0,1/2)}} h \leq C_H \inf_{\partial B(0,1/2)} h,$$

where the latter inequality follows from the harmonicity of h and the maximum principle.

Now, we define in $\overline{B(0,1)}\backslash\{0\}$, the function

$$v(x) = \begin{cases} -\ln(|z|) & \text{if} \quad N = 2, \\ \frac{1}{|z|^{N-2}} - 1 & \text{if} \quad N \geq 3, \end{cases}$$

and we recall that v solves

$$\begin{cases} -\Delta v = 0 & \text{in} \quad B(0,1)\backslash\{0\}, \\ v = 0 & \text{in} \quad \partial B(0,1). \end{cases} \tag{2.5.48}$$

Suppose $N \geq 3$. If $y \in \partial B(0,1/2)$, then

$$h(y) \geq \inf_{\partial B(0,1/2)} h \geq \frac{h(0)}{C_H} \frac{v(y)}{2^{N-2}-1} = \frac{h(0)}{C_H(2^{N-2}-1)} v(y). \tag{2.5.49}$$

We let $\varepsilon = \frac{h(0)}{C_H(2^{N-2}-1)} > 0$, $A = B(0,1) \backslash \overline{B(0,1/2)}$ and we consider the function

$$w(z) = h(z) - \varepsilon v(z), \qquad \forall\, z \in A.$$

Hence, (2.5.47), (2.5.48) and (2.5.49) ensure that

$$\begin{cases} -\Delta w = 0 & \text{in} \quad A, \\ w \geq 0 & \text{on} \quad \partial A, \end{cases}$$

and the maximum principle yields

$$w \geq 0 \quad \text{in} \quad A.$$

Since $w(x) = 0$, we have

$$\frac{\partial h}{\partial \eta}(x) - \varepsilon \frac{\partial v}{\partial \eta}(x) = \frac{\partial w}{\partial \eta}(x) \leq 0. \tag{2.5.50}$$

However, here $\eta = x$, thus

$$\frac{\partial v}{\partial \eta}(x) = (\nabla v(x) \cdot x) = 2 - N. \tag{2.5.51}$$

Finally, thanks to (2.5.50) and (2.5.51), we obtain

$$\left|\frac{\partial h}{\partial \eta}(x)\right| \geq -\frac{\partial h}{\partial \eta}(x) \geq \frac{N-2}{C_H(2^{N-2}-1)} h(0) > 0. \tag{2.5.52}$$

Now, for any $j = 1, ..., N$, by the mean value Theorem for harmonique functions, and the positivity of h, we have

$$\left|\frac{\partial h}{\partial z_j}(0)\right| = \left|\frac{1}{\omega_N}\int_{B(0,1)} \frac{\partial h}{\partial z_j} dz\right| = \left|\frac{1}{\omega_N}\int_{\partial B(0,1)} h\eta_j d\sigma\right| \leq \frac{\alpha_N}{\omega_N}\frac{1}{\alpha_N}\int_{\partial B(0,1)} h d\sigma = Nh(0),$$

where ω_N denotes the measure of the unit ball $B(0,1)$ and α_N is the measure of $\partial B(0,1)$.

From the latter and (2.5.52) we infer that

$$|\nabla h(0)| \leq C(N)\left|\frac{\partial h}{\partial \eta}(x)\right|.$$

This proves the Lemma when $N \geq 3$.

For $N = 2$, we observe that for any $y \in \partial B(0,1/2)$,

$$h(y) \geq \inf_{\partial B(0,1/2)} h \geq \frac{h(0)}{C_H} \frac{v(y)}{\ln(2)} = \frac{h(0)}{C_H \ln(2)} v(y), \tag{2.5.53}$$

then the claim follows by the same procedure above, with $\varepsilon = \frac{h(0)}{\ln(2) C_H} > 0$. $\square$

Lemma 2.5.5. *Let $N \geq 2$, $U \subset \mathbb{R}^N$ be a domain and $u \in C^1(U)$ be a solution of*

$$-\Delta u = f \quad in \quad \mathcal{D}'(U), \tag{2.5.54}$$

with $f \in L^\infty_{loc}(U)$.
Assume $\delta > 0$ and $y \in U$ such that $\overline{B(y,\delta)} \subset U$. Then, for any $x \in B(y, \frac{\delta}{2})$ we have

$$|u(x) - u(y)| \leq \frac{\sqrt{N}}{2^{1-\gamma}}\Big(2N\|u\|_{L^\infty(\overline{B(y,\delta)})}\delta^{-\gamma} + \|f\|_{L^\infty(\overline{B(y,\delta)})}\delta^{2-\gamma}\Big)|x-y|^\gamma, \tag{2.5.55}$$

with $\gamma \in]0,1]$.

Proof. Let $y \in U$ and $R > 0$ such that $\overline{B(y,R)} \subset U$ then, for any $j = 1, ..., N$, we have

$$\Big|\frac{\partial u}{\partial x_j}(y)\Big| \leq \Big(\frac{N}{R}\|u\|_{L^\infty(\partial B(y,R))} + \frac{R}{2}\|f\|_{L^\infty(\overline{B(y,R)})}\Big). \tag{2.5.56}$$

This is exactly the same as the estimate (3.15) on page 38 of [GT01]. Indeed, since u is C^1, the barrier function Ψ is smooth and f belongs to $L^\infty_{\text{loc}}(U)$, we can apply the weak maximum principle (see, for example, Theorem 8.1 in [GT01]) to complete the proof in [GT01] and thus obtain the gradient estimate above (2.5.56).

By the mean value theorem, for any $\gamma \in]0,1]$,

$$|u(x) - u(y)| \leq \sup_{z \in \overline{B(y,\frac{\delta}{2})}} |\nabla u(z)||x-y| \leq \frac{\delta^{1-\gamma}}{2^{1-\gamma}} \sup_{z \in \overline{B(y,\frac{\delta}{2})}} |\nabla u(z)||x-y|^\gamma. \tag{2.5.57}$$

Moreover, for any $z \in \overline{B(y, \frac{\delta}{2})}$ we have $\overline{B(z, \frac{\delta}{2})} \subset \overline{B(y,\delta)} \subset U$, so we can apply (2.5.56) (with $R = \frac{\delta}{2}$) to get

$$|\nabla u(z)| \leq \sqrt{N}\Big(\frac{2N}{\delta}\|u\|_{L^\infty(\overline{B(y,\delta)})} + \frac{\delta}{4}\|f\|_{L^\infty(\overline{B(y,\delta)})}\Big), \tag{2.5.58}$$

for any $z \in \overline{B(y, \frac{\delta}{2})}$. The claim then follows by combining (2.5.57) and (2.5.58). □

We conclude with the following useful extension lemma (in which we keep an explicit track of all constants and their dependencies).

Lemma 2.5.6. *Assume $N \geq 2$, $\alpha \in (0,1]$ and $r > 0$. Let $g \in C^1(\mathbb{R}^{N-1})$ be such that $\nabla g \in C^{0,\alpha}_{loc}(\mathbb{R}^{N-1})$ and let us denote by Ω the epigraph defined by g.*

(i) For any $v \in C^{1,\alpha}(\overline{\Omega \cap (B'(0',r) \times \mathbb{R})})$ there exists $\widetilde{w} \in C^{1,\alpha}(\overline{B'(0',r) \times \mathbb{R}})$ such that

$$\widetilde{w} = v \quad in \quad \overline{\Omega \cap (B'(0',r) \times \mathbb{R})}$$

and

$$\|\widetilde{w}\|_{C^{1,\alpha}(\overline{B'(0',r)\times\mathbb{R}})} \leq C(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'(0',r)})})^4\|v\|_{C^{1,\alpha}(\overline{\Omega\cap(B'(0',r)\times\mathbb{R})})}$$

where $C = C(N)$ is a positive constant.

(ii) Assume $\widetilde{T} > \sup_{B'(0',r)} |g|$ and $v \in C^{1,\alpha}(\overline{\mathfrak{C}^g(0', r, \widetilde{T})})$.
Then, for any $\sup_{B'(0',r)} |g| < t < \widetilde{T}$ there exists $\widetilde{w_t} \in C^{1,\alpha}(\overline{B'(0', r) \times (-t, t)})$ such that

$$\widetilde{w_t} = v \quad in \quad \overline{\mathfrak{C}^g(0', r, t)}$$

and

$$\|\widetilde{w_t}\|_{C^{1,\alpha}(\overline{B'(0',r)\times(-t,t)})} \leq C(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'(0',r)})})^4 \|v\|_{C^{1,\alpha}(\overline{\mathfrak{C}^g(0',r,\widetilde{T})})}$$

where $C = C(N, t, \widetilde{T})$ is a positive constant.

Proof. To simplify the presentation, we set $B'_r = B'(0', r)$ and $\Omega_r = \Omega \cap (B'(0', r) \times \mathbb{R})$. Then we consider the following two functions :

$$\begin{array}{rccc} \psi : & \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ & x = (x', x_N) & \longmapsto & (x', x_N - g(x')) \end{array}$$

and

$$\begin{array}{rccc} \theta : & \mathbb{R}^N & \longrightarrow & \mathbb{R}^N \\ & y = (y', y_N) & \longmapsto & (y', y_N + g(y')) \end{array}$$

and we observe that $\psi^{-1} = \theta$, $\psi(\overline{\Omega_r}) = \overline{B'_r \times \mathbb{R}_+}$, $\psi(\partial\Omega \cap (B'_r \times \mathbb{R})) = \partial\mathbb{R}^N_+ \cap (B'_r \times \mathbb{R})$ and $\psi((B'_r \times \mathbb{R})\backslash\overline{\Omega_r}) = (B'_r \times \mathbb{R})\backslash\overline{\mathbb{R}^N_+}$. Moreover,

$$\begin{aligned} |\theta(x) - \theta(y)| &\leq |x - y| + |g(x') - g(y')| \leq (1 + \|\nabla g\|_{L^\infty(B'_r)})|x - y|, \quad \forall\, x, y \in \overline{B'_r \times \mathbb{R}}, \\ |\psi(x) - \psi(y)| &\leq |x - y| + |g(x') - g(y')| \leq (1 + \|\nabla g\|_{L^\infty(B'_r)})|x - y|, \quad \forall\, x, y \in \overline{B'_r \times \mathbb{R}}. \end{aligned} \tag{2.5.59}$$

(i) For any $y \in \overline{B'_r \times \mathbb{R}_+}$ set $\widetilde{v}(y) = v(\theta(y))$, then

$$\widetilde{v} \in C^{1,\alpha}(\overline{B'_r \times \mathbb{R}_+}).$$

Indeed, by definition, we have $\widetilde{v} \in C^0(\overline{B'_r \times \mathbb{R}_+})$ and

$$\|\widetilde{v}\|_{L^\infty(B'_r\times\mathbb{R}_+)} = \|v\|_{L^\infty(\Omega_r)} \tag{2.5.60}$$

moreover, for any $x, y \in B'_r \times \mathbb{R}_+$,

$$|\widetilde{v}(x) - \widetilde{v}(y)| \leq [v]_{C^{0,\alpha}(\Omega_r)} |\theta(x) - \theta(y)|^\alpha \leq [v]_{C^{0,\alpha}(\Omega_r)} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha |x - y|^\alpha, \tag{2.5.61}$$

where in the latter we have used (2.5.59). Hence,

$$[\widetilde{v}]_{C^{0,\alpha}(B'_r\times\mathbb{R}_+)} \leq [v]_{C^{0,\alpha}(\Omega_r)} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha. \tag{2.5.62}$$

Also, for any $i \in \{1, \cdots, N-1\}$,

$$\partial_i \widetilde{v}(y) = \partial_i v(\theta(y)) + \partial_i g(y')\partial_N v(\theta(y))$$

and

$$\partial_N \widetilde{v}(y) = \partial_N v(\theta(y)),$$

thus,

$$\|\nabla \widetilde{v}\|_{L^\infty(B'_r \times \mathbb{R}_+)} \leq \sqrt{2}(1 + \|\nabla g\|_{L^\infty(B'_r)})\|\nabla v\|_{L^\infty(\Omega_r)}. \tag{2.5.63}$$

Furthermore, for any $x, y \in B'_r \times \mathbb{R}_+$ and for any $i \in \{1, \cdots, N-1\}$, we have

$$\begin{aligned}
|\partial_i \widetilde{v}(x) - \partial_i \widetilde{v}(y)| &\leq |\partial_i v(\theta(x)) - \partial_i v(\theta(y))| + |\partial_i g(x')\partial_N v(\theta(x)) - \partial_i g(y')\partial_N v(\theta(y))| \\
&\leq [\partial_i v]_{C^{0,\alpha}(\overline{\Omega_r})} |\theta(x) - \theta(y)|^\alpha + |\partial_i g(x') - \partial_i g(y')| \partial_N v(\theta(x))| \\
&+ |\partial_i g(y')||\partial_N v(\theta(x)) - \partial_N v(\theta(y))| \leq [\partial_i v]_{C^{0,\alpha}(\overline{\Omega_r})} |\theta(x) - \theta(y)|^\alpha \\
&+ |\partial_N v(\theta(x))| [\partial_i g]_{C^{0,\alpha}(\overline{B'_r})} |x' - y'|^\alpha + |\partial_i g(y')| [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r}))} |\theta(x) - \theta(y)|^\alpha \\
&\leq [\partial_i v]_{C^{0,\alpha}(\overline{\Omega_r})} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha |x - y|^\alpha + \|\partial_N v\|_{L^\infty(\Omega_r)} [\partial_i g]_{C^{0,\alpha}(\overline{B'_r})} |x' - y'|^\alpha \\
&+ \|\partial_i g\|_{L^\infty(B'_r)} [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r})} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha |x - y|^\alpha \\
&\leq \Big(([\partial_i v]_{C^{0,\alpha}(\overline{\Omega_r})} + \|\partial_i g\|_{L^\infty(B'_r)} [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r})})(1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha\Big) |x - y|^\alpha + \\
&+ \|\partial_N v\|_{L^\infty(\Omega_r)} [\partial_i g]_{C^{0,\alpha}(\overline{B'_r})} |x - y|^\alpha \\
&\leq (1 + \|\nabla g\|_{L^\infty(B'_r)})^{\alpha+1}([\partial_i v]_{C^{0,\alpha}(\overline{\Omega_r})} + [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r})})|x - y|^\alpha + \\
&+ \|\partial_N v\|_{L^\infty(\Omega_r)} [\partial_i g]_{C^{0,\alpha}(\overline{B'_r})} |x - y|^\alpha \\
&\leq 2(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{\alpha+1} \|\nabla v\|_{C^{0,\alpha}(\overline{\Omega_r})} |x - y|^\alpha.
\end{aligned} \tag{2.5.64}$$

Hence,

$$[\partial_i \widetilde{v}]_{C^{0,\alpha}(\overline{B'_r \times \mathbb{R}_+})} \leq 2(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{\alpha+1} \|\nabla v\|_{C^{0,\alpha}(\overline{\Omega_r})}. \tag{2.5.65}$$

Similarly we have

$$|\partial_N \widetilde{v}(x) - \partial_N \widetilde{v}(y)| \leq [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r})} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha |x - y|^\alpha \tag{2.5.66}$$

and so

$$[\partial_N \widetilde{v}]_{C^{0,\alpha}(\overline{B'_r \times \mathbb{R}_+})} \leq [\partial_N v]_{C^{0,\alpha}(\overline{\Omega_r})} (1 + \|\nabla g\|_{L^\infty(B'_r)})^\alpha. \tag{2.5.67}$$

Thanks to (2.5.60), (2.5.62), (2.5.63),(2.5.65) and (2.5.67) we deduce that $\widetilde{v} \in C^{1,\alpha}(\overline{B'_r \times \mathbb{R}_+})$ and

$$\|\widetilde{v}\|_{C^{1,\alpha}(\overline{B'_r \times \mathbb{R}_+})} \leq (2N + 1)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{\alpha+1} \|v\|_{C^{1,\alpha}(\overline{\Omega_r})}. \tag{2.5.68}$$

Now, for any $y \in \overline{B'_r \times \mathbb{R}}$ we set

$$\widetilde{V}(y) = \begin{cases} \widetilde{v}(y', y_N) & \text{if} \quad y_N \geq 0, \\ \displaystyle\sum_{i=1}^{2} c_i \widetilde{v}\left(y', -\frac{y_N}{i}\right) & \text{if} \quad y_N < 0, \end{cases}$$

where $c_1 = -3$ and $c_2 = 4$. With this choice, we see that $\widetilde{V}$ belongs to $C^1(B'_r \times \mathbb{R})$ and

$$\|\widetilde{V}\|_{L^\infty(B'_r\times\mathbb{R})} + \|\nabla\widetilde{V}\|_{L^\infty(B'_r\times\mathbb{R})} \le 7(N+1)\|\widetilde{v}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}_+})}. \tag{2.5.69}$$

Now, using the value of c_1 and c_2, and

$$|x_N + \frac{y_N}{i}| \le |x_N| + \frac{|y_N|}{i} \le |x_N| + |y_N| \le x_N - y_N = |x_N - y_N|,$$

it follows that, for any $i \in \{1, \cdots, N\}$

$$\left[\partial_i \widetilde{V}\right]_{C^{0,\alpha}(\overline{B'_r\times\mathbb{R}})} \le \; C_1(N)\|\widetilde{v}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}_+})}. \tag{2.5.70}$$

Hence, (2.5.68), (2.5.69) and (2.5.70) lead to

$$\|\widetilde{V}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}})} \le C_2(N)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{\alpha+1}\|v\|_{C^{1,\alpha}(\overline{\Omega_r})}. \tag{2.5.71}$$

To proceed further, for any $y \in \overline{B'_r \times \mathbb{R}}$ set $\widetilde{w}(y) := \widetilde{V}(\psi(y))$, then $\widetilde{w}$ satisfies the claim. Indeed, if $y \in \overline{\Omega_r}$ then $\psi(y) \in \overline{B'_r \times \mathbb{R}_+}$, and so $\widetilde{V}(\psi(y)) = \widetilde{v}(\psi(y))$. Hence, $\widetilde{w}(y) = \widetilde{V}(\psi(y)) = \widetilde{v}(\psi(y)) = v((\theta \circ \psi)(y)) = v(y)$ for any $y \in \overline{\Omega_r}$. Also, in view of the definition of ψ and (2.5.59), the same computations leading to (2.5.68) yield

$$\|\widetilde{w}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}})} \le (2N+1)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{\alpha+1}\|\widetilde{V}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}})}. \tag{2.5.72}$$

By putting together (2.5.72) and (2.5.71) we get

$$\|\widetilde{w}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}})} \le C_3(N)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^{2(\alpha+1)}\|v\|_{C^{1,\alpha}(\overline{\Omega_r})}. \tag{2.5.73}$$

This concludes the proof of the first part of the Lemma.

(ii) Let us consider an even function $\eta_t \in C^2_c(\mathbb{R})$ such that $0 \le \eta_t \le 1$,

$$\eta_t(x_N) := \begin{cases} 1 & \text{if} \quad 0 \le x_N \le t, \\ 0 & \text{if} \quad x_N \ge t + \frac{\widetilde{T}-t}{2}. \end{cases}$$

Then, $v\eta_t \in C^{1,\alpha}(\overline{\Omega_r})$ and $\|v\eta_t\|_{C^{1,\alpha}(\overline{\Omega_r})} = \|v\eta_t\|_{C^{1,\alpha}(\overline{\mathfrak{C}^g(0',r,\widetilde{T})})} \le C(\eta_t)\|v\|_{C^{1,\alpha}(\overline{\mathfrak{C}^g(0',r,\widetilde{T})})}$.

From the first part of Lemma we get $\widetilde{v\eta_t} \in C^{1,\alpha}(\overline{B'_r \times \mathbb{R}})$ such that $\widetilde{v\eta_t} = v\eta_t$ on $\overline{\Omega_r}$ and

$$\begin{aligned} \|\widetilde{v\eta_t}\|_{C^{1,\alpha}(\overline{B'_r\times\mathbb{R}})} &\le C_3(N)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^4\|v\eta_t\|_{C^{1,\alpha}(\overline{\Omega_r})} \\ &\le C(\eta_t)C_3(N)(1 + \|\nabla g\|_{C^{0,\alpha}(\overline{B'_r})})^4\|v\|_{C^{1,\alpha}(\overline{\mathfrak{C}^g(0',r,\widetilde{T})})} \end{aligned}$$

To conclude, we set $\widetilde{w} = \widetilde{v\eta_t}_{|\overline{B'_r\times(-t,t)}}$ and we observe that, the positive constant $C(\eta_t)$ depends only on t and $\widetilde{T}$. $\square$

Chapter 3

One-dimensional symmetry results for semilinear equations and inequalities on half-spaces

Contents

3.1 Introduction

In this work we consider solutions, not necessarily bounded, to the elliptic boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{3.1.1}$$

where $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid x_N > 0\}$ is the open upper half-space of $\mathbb{R}^N$, $N \geq 2$ and f is a locally Lipschitz-continuous function on $[0, +\infty)$.
Studying and classifying solutions to problem (3.1.1) is natural for various reasons:

(i) the half-space is the simplest unbounded domain with an unbounded boundary in which one can study the Dirichlet problem for the semilinear Poisson equation.

(ii) Problem (3.1.1) arises as a limit problem, after a blow-up procedure near the boundary, either when one wants to derive *a priori* estimates for positive solutions to some subcritical semilinear uniformly elliptic BVPs on smooth bounded domains ([GS81a]), or when one studies uniqueness results for classical solutions to the singular perturbation problem

$$-\varepsilon^2\Delta v = f(v) \quad \text{in} \quad \Omega, \qquad v > 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega, \tag{3.1.2}$$

where Ω is a smooth and bounded domain, f is a suitable smooth function and ε is a small and positive parameter (see for instance [Ang85] and [Dan09]).

Notice that, when $f(0) < 0$, even if solutions v considered in (3.1.2) are positive, given the absence of the strong maximum principle, the solution u of (3.1.1) obtained after the scaling procedure is "only" non-negative in $\mathbb{R}^N_+$.

This motivates the choice to consider the non-negative solutions of (3.1.1) (and not only the positive ones).

Finally, the need to also consider unbounded solutions to (3.1.1) is well illustrated by the case of harmonic functions, that is when $f \equiv 0$. In this case, it is well-known that all the solutions to (3.1.1) are given by $u(x) = \alpha x_N$, $\alpha \geq 0$.

The first result concerning the complete classification of the solutions to (3.1.1) with $f \not\equiv 0$ was obtained by Gidas and Spruck in 1981. It concerns the function $f(u) = u^p$ and it was motivated by the the first situation described in item (ii) above.

Theorem A ([GS81a]) *Assume $N \geq 2$ and let p be any real number satisfying $1 < p \leq \frac{N+2}{N-2}$. The only classical solution to*

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

is $u \equiv 0$.

The proof in [GS81a] makes use of the moving spheres method. In 1993, Berestycki, Caffarelli and Nirenberg ([BCN93]) observed that the proof in [GS81a] actually yields the following result

Theorem B ([BCN93]) *Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of (3.1.1) where $f \in Lip_{loc}([0, +\infty))$. If any of the two following conditions holds:*

(i) $N = 2$ and $f \geq 0$,

(ii) $N \geq 3$ and f satisfies

$$t \mapsto \frac{f(t)}{t^{(N+2)/(N-2)}} \quad \textit{is nonincreasing on } (0, +\infty). \tag{3.1.3}$$

Then, either $u \equiv 0$, or there exists a function $u_0 : [0, +\infty) \to [0, +\infty)$ such that

$$u(x) = u_0(x_N) \qquad \textit{for any } x \in \mathbb{R}^N_+ \tag{3.1.4}$$

and $u_0 > 0$, $u_0' > 0$ on $(0, +\infty)$.

Solutions of the form (3.1.4), that is, solutions depending only on the variable x_N, are called *one-dimensional* since they inherit the symmetry of the underlying domain $\mathbb{R}^N_+$. Clearly, the profile u_0 solves the one-dimensional problem

$$\begin{cases} -u_0'' = f(u_0) \quad \text{in} \quad (0, +\infty), \\ u_0(0) = 0. \end{cases} \tag{3.1.5}$$

Note that we necessarily have $f(0) \geq 0$ in the above Theorem B. Recently, the following results about the one-dimensional symmetry of the solutions to (3.1.1) have been obtained. They also cover the case in which $f(0) < 0$.

Theorem C ([FS16; FS17]) *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution of* (3.1.1) *where $f \in Lip_{loc}([0, +\infty))$. Then,*

(i) if $f(0) \geq 0$, then either $u \equiv 0$, or u is positive on $\mathbb{R}^2_+$ and strictly increasing in the direction orthogonal to the boundary, i.e., $\frac{\partial u}{\partial x_2} > 0$ on $\mathbb{R}^2_+$;

(ii) if $f(0) < 0$, then either $u > 0$ and $\frac{\partial u}{\partial x_2} > 0$ on $\mathbb{R}^2_+$, or

$$u(x) = u_0(x_2) \qquad \textit{for any } x \in \mathbb{R}^2_+$$

for a unique, periodic function $u_0 : [0, +\infty) \to [0, +\infty)$;

(iii) if for some $p > 1$,

$$\lim_{t \to +\infty} \frac{f(t)}{t^p} \in (0, +\infty), \tag{3.1.6}$$

then u is bounded and one-dimensional.[1]

(iv) if $f \in C^1([0, +\infty))$ with $f(0) < 0$ and $f'(t) \geq \alpha > 0$ for any $t > 0$, then u is one-dimensional and periodic.

Theorem D *Assume $N \geq 2$. The only classical solution to*

$$\begin{cases} -\Delta u = u - 1 & \textit{in} \quad \mathbb{R}^N_+, \\ u \geq 0 & \textit{in} \quad \mathbb{R}^N_+, \\ u = 0 & \textit{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

is the function

$$u(x) = 1 - \cos(x_N) \quad \textit{for any } x \in \mathbb{R}^N_+.$$

The case $N = 2$ in Theorem D is due to [FS16] (just apply item (iv) of Theorem C with $f(u) = u - 1$), while the case $N \geq 3$ is proven in [CEG16].

It is interesting to note that the one-dimensional symmetry results in the four theorems above were obtained without any additional assumption on the non-negative solution u. It therefore seems natural to seek to broaden the class of functions f guaranteeing that

[1] more precisely, $u(x) = u_0(x_2)$ for any $x \in \mathbb{R}^2_+$ and for some bounded function $u_0 : [0, +\infty) \to [0, +\infty)$ solving (3.1.5). Moreover, either u_0 is periodic (possibly identically equal to zero) or, $u_0 > 0$ and $u_0' > 0$ on $(0, +\infty)$.

the only classical solutions, bounded or not, to (3.1.1) are necessarily one-dimensional. On the other hand, such a one-dimensional symmetry result is not true for every locally Lipschitz-continuous function f, as shown by the function $u(x) = x_N e^{-x_1}$, which solves (3.1.1) with $f(u) = -u$ and $N \geq 2$. Hence, the following two research lines naturally emerge :

(1) provide general natural conditions on f guaranteeing that the only classical solutions, bounded or not, to (3.1.1) are necessarily one-dimensional;

(2) supply sufficiently general assumptions on classical solutions to (3.1.1) ensuring their one-dimensional symmetry (either independently of the considered function $f \in Lip_{\rm loc}([0, +\infty))$, or for an appropriate subclass of locally continuous Lipschitz functions).

The new results presented in this article are devoted to both lines of research. Before stating our results, let us review those that exist in the literature and that relate to research line (2) above.

With regard to the second line of research, various additional hypotheses on u have been considered in the literature in order to establish their one-dimensional symmetry. They mainly concern the growth conditions of u or its qualitative or spectral properties, such as monotonicity, stability and/or the Morse index. These conditions are natural, as they are generally satisfied by the solutions that appear in applications and by those constructed using variational or topological methods. In this regard, we present below the main contributions found in the literature. We shall describe them in ascending order of generality. Let us begin with the case of bounded solutions.

Theorem E ([BCN97a]) *Assume $N = 2$ or 3. Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a positive and bounded solution to* (3.1.1) *where $f \in Lip_{loc}([0, +\infty))$.*
If $N = 2$, u is one-dimensional and strictly increasing. If $N = 3$ the same conclusion holds, if one assumes in addition that $f(0) \geq 0$ and that f is C^1.

More recently, by combining Theorem C above and a geometric tool introduced in [FV10], the authors of [FS16] improved the result of Theorem E in the two-dimensional case. Indeed, the authors of [FS16] prove the one-dimensional symmetry for any non-negative solution and assuming only that u has a bounded gradient. More precisely,

Theorem F ([FS16]) *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution to* (3.1.1) *where $f \in Lip_{loc}([0, +\infty))$. If u has bounded gradient, then it is one-dimensional.*

Recall that, bounded solutions of (3.1.1) have bounded gradient, but the converse is not true, as shown by the (one-dimensional) harmonic function $u(x) = x_N$. Furthermore, Theorem F also covers the case of one-dimensional periodic solutions which, given the homogeneous Dirichlet condition, must necessarily vanish somewhere in $\mathbb{R}^2_+$.[2]

To conclude on the case of bounded solutions of (3.1.1) we mention the two following results. The first one deals with convex functions f with $f(0) = 0$, while the second one applies, among other things, to any non-negative function f.

[2]This fact is well illustrated by the explicit example discussed in Theorem D.

Theorem G ([CLZ14]) *Assume $N \geq 2$ and let $f \in C^1([0,+\infty)) \cap C^2((0,+\infty))$ with $f(0)=0$ and $f'' \geq 0$. The only bounded classical solution to* (3.1.1) *is $u \equiv 0$.*

As an immediate consequence of Theorem G one has

Corollary A ([CLZ14]) *Assume $N \geq 2$ and let $p > 1$. The only bounded classical solution to*

$$\begin{cases} -\Delta u = u^p & in \quad \mathbb{R}^N_+, \\ u \geq 0 & in \quad \mathbb{R}^N_+, \\ u = 0 & on \quad \partial\mathbb{R}^N_+, \end{cases}$$

is $u \equiv 0$.

Corollary A was first proven for $1 < p < (N+1)/(N-3)^+$ in [Dan92], and then for any $1 < p < p_{JL}(N)$ in [Far07], where p_{JL} is the Joseph-Lundgren stability exponent given by $p_{JL}(N) = \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)^+}$.

Theorem H ([DF22]) *Let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a bounded solution of* (3.1.1)*. Assume $f \in C^1([0,+\infty))$ and*

1. *either $f(t) \geq 0$ for $t \geq 0$,*
2. *or there exists $z > 0$ such that $f(t) \geq 0$ for $t \in [0,z]$ and $f(t) \leq 0$ for $t \geq z$.*

If $2 \leq N \leq 11$, then either $u \equiv 0$, or u is positive, one-dimensional and strictly increasing.

To continue, recall that a solution to (3.1.1) that is either strictly monotone in the direction x_N or a local minimizer is automatically *stable*, which means that it additionally satisfies

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 - f'(u)\varphi^2 \geq 0, \tag{3.1.7}$$

for all $\varphi \in C^1_c(\mathbb{R}^N_+)$ (here we have supposed $f \in C^1$).
Also, we shall say that u is *stable outside a compact* $\mathcal{K} \subset \mathbb{R}^N_+$ if the integral inequality (3.1.7) holds for any $\varphi \in C^1_c(\mathbb{R}^N_+ \setminus \mathcal{K})$. Clearly, the stability implies the stability outside a compact set (actually, any finite Morse index solution to (3.1.1) is stable outside a compact set of $\mathbb{R}^N_+$).

As for these classes of solutions to (3.1.1), we have

Theorem I ([DSS22]) *Assume $N \geq 2$. Let $f \in C^1([0,+\infty)) \cap C^2((0,+\infty))$ be non-negative and convex, with $f(0)=0$ and $f \not\equiv 0$.*
Let u be a classical solution of (3.1.1) *which is monotone in the x_N direction, i.e. such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Then, $u \equiv 0$.*
In particular, $u \equiv 0$ is the only classical solution to (3.1.1) *which is bounded on finite strips.*[3]

As an immediate consequence of Theorem I one has

[3] When $f(0) \geq 0$, any positive solution to (3.1.1), which is also bounded on finite strips, is strictly monotone in the x_N-direction (see Theorem 3.1 of [**far1**]).

Corollary B ([DSS22]) *Assume $N \geq 2$ and let $p > 1$. Let u be a classical solution of*

$$\begin{cases} -\Delta u = u^p & in \quad \mathbb{R}^N_+, \\ u \geq 0 & in \quad \mathbb{R}^N_+, \\ u = 0 & on \quad \partial\mathbb{R}^N_+, \end{cases} \tag{3.1.8}$$

which is monotone in the x_N direction. Then, $u \equiv 0$.
In particular, $u \equiv 0$ is the only classical solution to (3.1.8) *which is bounded on finite strips.*

Corollary B recovers and improves upon Corollary A above.

Very recently, Corollary B has been improved in the work [DFP23]. Specifically,

Theorem J ([DFP23]) *Assume $N \geq 2$ and let $p > 1$. The only classical solution to*

$$\begin{cases} -\Delta u = |u|^{p-1}u & in \quad \mathbb{R}^N_+, \\ u = 0 & on \quad \partial\mathbb{R}^N_+, \end{cases}$$

which is stable outside a compact set of $\mathbb{R}^N_+$ is $u \equiv 0$.

Note that Theorem J holds for sign-changing solutions. It was first proven in [Far07] for all $N \leq 10$ and, if $N > 10$, for $1 < p < p_{JL}(N) = \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}$, the Joseph-Lundgren stability exponent.

We are now ready to present our results. The first theorem provides a contribution to the first line of research above. It recovers and improves upon the result stated in item (iii) of the previous Theorem C. It only requires that f behaves at least linearly at infinity, i.e., that f satisfies (3.1.9) below.

Theorem 3.1.1. *Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a solution of* (3.1.1) *where $f \in Lip_{loc}([0, +\infty))$ satisfies*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{3.1.9}$$

Then there exists a bounded function $u_0 : [0, +\infty) \to [0, +\infty)$ such that

$$u(x) = u_0(x_2) \quad \text{for any } x \in \mathbb{R}^2_+.$$

Moreover, either u_0 is periodic (possibly identically equal to zero) or, $u_0 > 0$ and $u_0' > 0$ on $(0, +\infty)$.

Next we state our results concerning the second line of research above. Let us begin with the case of solutions monotone in the x_N-direction.

Theorem 3.1.2. *Assume $2 \leq N \leq 9$ and let $u \in C^2(\overline{\mathbb{R}^N_+})$ be a solution of* (3.1.1) *such that $\frac{\partial u}{\partial x_N} \geq 0$ on $\mathbb{R}^N_+$. Assume that $f \in C^1([0, +\infty))$ is a non-negative function satisfying*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{3.1.10}$$

Then, either $u \equiv 0$*, or there exists a bounded function* $u_0 : [0, +\infty) \to [0, +\infty)$ *such that*

$$u(x) = u_0(x_N) \qquad \text{for any } x \in \mathbb{R}^N_+,$$

and $u_0 > 0$*,* $u_0' > 0$ *on* $(0, +\infty)$*.*

Moreover, if $f(t) > 0$ *for any* $t > 0$*, then necessarily* $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = 0$*.*

Although subject to a dimensional restriction, the previous result extends Theorem I to all non-negative functions f under the sole assumption that f behave at least linearly at infinity.[4] It would be interesting to know whether this result still holds for $N > 9$.

If we further assume that f is convex or that f has at most polynomial growth, the conclusions of Theorem 3.1.2 remain valid up to dimension $N = 10$. Specifically,

Theorem 3.1.3. *Assume* $2 \le N \le 10$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (3.1.1) *such that* $\frac{\partial u}{\partial x_N} \ge 0$ *on* $\mathbb{R}^N_+$*. Assume that* $f \in C^1([0, +\infty))$ *is a non-negative, convex function satisfying*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{3.1.11}$$

Then, either $u \equiv 0$*, or there exists a bounded function* $u_0 : [0, +\infty) \to [0, +\infty)$ *such that*

$$u(x) = u_0(x_N) \qquad \text{for any } x \in \mathbb{R}^N_+,$$

and $u_0 > 0$*,* $u_0' > 0$ *on* $(0, +\infty)$*.*

Moreover, if $f(0) = 0$*, then necessarily* $u \equiv 0$ *in* $\mathbb{R}^N_+$*.*

Unlike Theorem I, the function f in Theorem 3.1.3 is not necessarily non-decreasing. For instance, Theorem 3.1.3 applies to $f(t) = |t-1|^p$, $p > 1$.

We also note that, if we drop the assumption (3.1.11), then the full conclusion of Theorem 3.1.3 would no longer be valid. Indeed, when $f(t) = e^{-2t}$, which is positive and convex, the *unbounded* and monotone function $u(x) = \ln(1 + x_N)$ solves (3.1.1) for any $N \ge 2$.

This observation also applies to Theorem 3.1.2 above and to the following theorem.

Theorem 3.1.4. *Assume* $2 \le N \le 10$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (3.1.1) *such that* $\frac{\partial u}{\partial x_N} \ge 0$ *on* $\mathbb{R}^N_+$*. Assume that* $f \in C^1([0, +\infty))$ *is a non-negative function satisfying*

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0, \qquad \limsup_{t \to +\infty} \frac{f(t)}{t^\theta} < +\infty, \tag{3.1.12}$$

for some $\theta > 1$*.*

Then, either $u \equiv 0$*, or there exists a bounded function* $u_0 : [0, +\infty) \to [0, +\infty)$ *such that*

$$u(x) = u_0(x_N) \qquad \text{for any } x \in \mathbb{R}^N_+,$$

[4] Note that, the convex function f in Theorem I necessarily satisfies the assumption (3.1.10).

and $u_0 > 0$, $u_0' > 0$ *on* $(0, +\infty)$.

Moreover, if $f(t) > 0$ *for any* $t > 0$, *then necessarily* $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = 0$.

As an immediate consequence of previous three theorems we can treat the case of solutions that are bounded on finite strips.

Corollary 3.1.1. *Let* N, f *be as in the statement of Theorem 3.1.2 (resp. Theorem 3.1.3 or Theorem 3.1.4) and let* u *be a classical solution of* (3.1.1) *which is bounded on finite strips. Then, the conclusions of Theorem 3.1.2 (resp. Theorem 3.1.3 or Theorem 3.1.4) still hold true.*

The viewpoint we adopted to prove the previous theorems can also be applied to both stable solutions and stable solutions outside a compact set. Below we present the Liouville-type theorems corresponding to these two classes of solutions.

Theorem 3.1.5. *Assume* $2 \leq N \leq 4$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a stable solution of* (3.1.1) *where* $f \in C^1([0, +\infty))$ *is a non-negative and non-decreasing function satisfying*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{3.1.13}$$

Then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = f'(0) = 0$.

In the next result we replace the superlinearity condition (3.1.13) on f with a growth assumption on the solution u.

Theorem 3.1.6. *Assume* $2 \leq N \leq 4$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a stable solution of* (3.1.1) *where* $f \in C^1([0, +\infty))$ *is a non-negative, non-decreasing function with* $f \not\equiv 0$. *If*

$$u(x) = O(|x|^{\frac{4-N}{2}} \ln^{1/2} |x|) \qquad as \quad |x| \to +\infty, \tag{3.1.14}$$

then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = f'(0) = 0$.

The following two results deal with solutions which are stable outside a compact set of $\mathbb{R}^N_+$.

Theorem 3.1.7. *Assume* $2 \leq N \leq 9$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (3.1.1) *which is stable outside a compact set of* $\mathbb{R}^N_+$. *Let* $f \in C^1([0, +\infty))$ *be a non-negative, convex function with* $f(0) = 0$ *and* $f \not\equiv 0$.
Then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f'(0) = 0$.

Theorem 3.1.8. *Assume* $2 \leq N \leq 10$ *and let* $u \in C^2(\overline{\mathbb{R}^N_+})$ *be a solution of* (3.1.1) *which is stable outside a compact set of* $\mathbb{R}^N_+$, *where* $f \in C^1([0, +\infty))$ *is a non-negative and non-decreasing function satisfying*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty, \qquad \limsup_{t \to +\infty} \frac{f(t)}{t^\theta} < +\infty, \tag{3.1.15}$$

for some $\theta > 1$.
Then, $u \equiv 0$ *in* $\mathbb{R}^N_+$ *and* $f(0) = f'(0) = 0$.

In the case of monotone solutions, the flexibility of our approach allows us to extend some of the previous results to the case of differential inequalities, even without imposing any boundary conditions. More precisely, we have

Theorem 3.1.9. *Assume $N = 2, 3$ and let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u \geq f(u) & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \text{in } \mathbb{R}^N_+, \end{cases} \tag{3.1.16}$$

where $f \in C^0([0, +\infty))$ satisfies $f(t) > 0$ for any $t > 0$ and

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0.$$

Then, $u \equiv 0$ in $\mathbb{R}^N_+$ and $f(0) = 0$.

In particular, for $N = 2, 3$ and for any $p \geq 1$, the only non-negative and monotone solution to $-\Delta u \geq u^p$ in $\mathbb{R}^N_+$ is the function identically equal to zero.

We also note that the result above is sharp. Indeed, the function $u(x', x_N) = c(1 + |x'|^2)^{-\frac{1}{p-1}}$ satisfies $-\Delta u \geq u^p$ and $\frac{\partial u}{\partial x_N} \equiv 0$ in $\mathbb{R}^N_+$, when $N \geq 4$, $p > \frac{N-1}{N-3}$ and $c > 0$ is small enough. Nevertheless, we have the following sharp Liouville-type result.

Theorem 3.1.10. *Assume $N \geq 2$, let $p \geq 1$ be a real number and let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u \geq u^p & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \text{in } \mathbb{R}^N_+. \end{cases} \tag{3.1.17}$$

If $1 \leq p \leq \bar{p}(N)$, where

$$\bar{p}(N) := \begin{cases} +\infty & \text{if } N = 2, 3, \\ \frac{N-1}{N-3} & \text{if } N \geq 4, \end{cases} \tag{3.1.18}$$

then, $u \equiv 0$ in $\mathbb{R}^N_+$.

If we restrict ourselves to solutions of the semilinear Poisson equation on $\mathbb{R}^N_+$, but still without imposing boundary conditions, the classification results above can be extended to dimension $N \geq 4$. This is the content of the following two theorems.

Theorem 3.1.11. *Assume $2 \leq N \leq 9$ and let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \text{in } \mathbb{R}^N_+, \end{cases} \tag{3.1.19}$$

where $f \in C^1([0, +\infty))$ satisfies $f(t) > 0$ for any $t > 0$ and

$$\liminf_{t \to +\infty} \frac{f(t)}{t} > 0. \tag{3.1.20}$$

Then, either $u \equiv 0$ *in* $\mathbb{R}^N_+$, *or there exists a function* $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ *such that*

$$u(x) = u_0(x_1, ..., x_{N-1}) \qquad \text{for any } x \in \mathbb{R}^N_+.$$

If we further assume (3.1.12), *the above conclusion holds true even in dimension* $N = 10$.

In the case of the Lane-Emden equation $-\Delta u = u^p$, the above result can be improved as follows

Theorem 3.1.12. *Assume* $N \geq 2$ *and let* $u \in C^2(\mathbb{R}^N_+)$ *be a solution of*

$$\begin{cases} -\Delta u = u^p & \text{in } \ \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \ \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & \text{in } \ \mathbb{R}^N_+. \end{cases} \tag{3.1.21}$$

If $N = 2, 3$, *then* $u \equiv 0$ *in* $\mathbb{R}^N_+$*; while for* $N \geq 4$ *we have*

(i) if $1 \leq p < \frac{N+1}{N-3}$, *then* $u \equiv 0$ *in* $\mathbb{R}^N_+$*;*

(ii) if $p = \frac{N+1}{N-3}$, *then either* $u \equiv 0$ *in* $\mathbb{R}^N_+$ *or*

$$u(x) = (a + b|x' - x_0'|^2)^{-\frac{N-3}{2}} \qquad \text{for any } x = (x', x_N) \in \mathbb{R}^N_+, \tag{3.1.22}$$

for $a, b > 0$ *satisfying* $1 = ab(N-1)(N-3)$ *and any* $x_0' \in \mathbb{R}^{N-1}$*;*

(iii) if $\frac{N+1}{N-3} < p < p_{JL}(N-1)$, *where* $p_{JL}(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)^+}$ *is the Joseph-Lundgren stability exponent, then*
either $u \equiv 0$ *in* $\mathbb{R}^N_+$, *or there exists a function* $u_0 : \mathbb{R}^{N-1} \to (0, +\infty)$ *such that*

$$u(x) = u_0(x') \qquad \text{for any } x = (x', x_N) \in \mathbb{R}^N_+,$$

When $N \geq 4$, for any $\frac{N+1}{N-3} < p < p_{JL}(N-1)$, there exists a positive solution to (3.1.21) depending only on the first $N-1$ variables.[5] Also note that, the positive solutions in items (*ii*) and (*iii*) are unstable (see Theorem 5 and Remark 4 of [Far07]). Finally observe that, for $N \leq 11$, the result above provides the complete classification of the solutions to (3.1.21) for any $p \geq 1$.

3.2 Proofs of results in the case of the semi-linear Poisson equation

Proof of Theorem 3.1.1. By the results in [FS16; FS17] (see Theorem C in the Introduction), either u is one-dimensional and periodic (possibly identically equal to zero) and we

[5] For any $N \geq 3$ and $p > \frac{N+2}{N-2}$ positive radial solutions to $-\Delta u = u^p$ in $\mathbb{R}^N$ always exist (see for instance [GS81b],[JL73]). If u_0 is such a function in dimension $N-1$, then the function $u(x) = u_0(x')$, $x = (x', x_N) \in \mathbb{R}^N_+$, provides the desired example.

are done, or $u > 0$ and $\frac{\partial u}{\partial x_2} > 0$ in $\mathbb{R}^2_+$. In the latter case, by Lemma 3.4.2 the profile of u at infinity defined by

$$\overline{u}(x_1) := \lim_{x_2 \to +\infty} u(x_1, x_2), \qquad \text{for any } x_1 \in \mathbb{R},$$

is a non-negative member of $L^1_{loc}(\mathbb{R})$ such that $f(\overline{u}) \in L^1_{loc}(\mathbb{R})$ and

$$-\overline{u}'' \geq f(\overline{u}) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}). \tag{3.2.1}$$

Furthermore, the sequence of functions $(u_n)_{n \geq 1}$ defined by

$$u_n(x) := \tilde{u}(x_1, x_2 + n), \qquad \text{for any } x \in \mathbb{R}^2,$$

converges to $\overline{u}$ in $L^1_{loc}(\mathbb{R}^2)$.

From (3.2.1) and (3.1.9) we see that

$$-\overline{u}'' \geq A\overline{u} - B \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}), \tag{3.2.2}$$

for some constants $A > 0$ and $B \geq 0$. Then the function $v(x_1) = -\overline{u}(x_1) + \frac{B}{2}x_1^2$ belongs to $L^1_{loc}(\mathbb{R})$ and solves

$$-v'' \leq 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}), \tag{3.2.3}$$

hence, v is subharmonic in the sense of the Classical Potential Theory (see for instance Theorem 3.2.1 in [DV24] or Theorem 3.2.11 in [Hör94]). Then v is convex on $\mathbb{R}$ (see for instance Lemma 3.2.9 in [DV24]) thus continuous on $\mathbb{R}$ and so is $\overline{u}$. We observe that, for any compact set $K \subset \mathbb{R}^2$ we have $0 \leq u_n \leq \overline{u} \leq \|\overline{u}\|_{L^\infty(K)} \mathbf{1}_K$, since $\frac{\partial u}{\partial x_2} > 0 \in \mathbb{R}^2_+$, so $|f(\overline{u}) - f(u_n)| \leq L|\overline{u} - u_n|$ on K, where L is the Lipschitz constant of f in the compact interval $[0, \|\overline{u}\|_{L^\infty(K)}]$. Then, $f(u_n) \longrightarrow f(\overline{u})$ in $L^1_{loc}(\mathbb{R}^2)$ so that we can pass to the limit in the equation satisfied by u_n to get that

$$-\overline{u}'' = f(\overline{u}) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}). \tag{3.2.4}$$

Since $f(\overline{u})$ is continuous, from (3.2.4) we infer that $\overline{u} \in C^2(\mathbb{R})$ is, indeed, a classical solution. From the latter property and (3.2.2) we see that $\overline{u}$ is bounded[6] on $\mathbb{R}$ which, in turns, implies that also u is bounded on $\mathbb{R}^2_+$. Then, from Theorem E (or Theorem F) in the Introduction, it follows that u is one-dimensional. □

Proof of Theorem 3.1.2. By the strong maximum principle, either $u \equiv 0$ (and we are done) or $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. The latter implies that u is stable on $\mathbb{R}^N_+$ and so Theorem 1.2 in [Cab+20] tell us that

$$\|u\|_{L^\infty(B(x, \frac{R_0}{2}))} \leq C\|u\|_{L^1(B(x,R_0))} \qquad \text{for any} \quad B(x, R_0) \subset\subset \mathbb{R}^N_+, \tag{3.2.5}$$

where $C = C(N, R_0) > 0$ is constant.

The latter and the assumption (3.1.10) enable us to apply item (i) of Lemma 3.4.1 to

[6] Indeed, $\overline{u}$ satisfies $\frac{(\overline{u}'(t))^2}{2} + F(\overline{u}(t)) = \frac{(\overline{u}'(0))^2}{2} + F(\overline{u}(0))$, for any $t \in \mathbb{R}$, where $F(u) := \int_0^u f(s)ds$. Hence, $A\frac{(\overline{u}(t))^2}{2} - B\overline{u}(t) \leq \frac{(\overline{u}'(0))^2}{2} + F(\overline{u}(0))$. The boundedness of $\overline{u}$ then follows, since $A > 0$.

deduce that u is bounded on the half-space $\{x \in \mathbb{R}^N_+ : x_N > 2R_0\}$. The monotonicity of u then implies that u is bounded on $\mathbb{R}^N_+$. The desired conclusion the follows by invoking Theorem 4 in [DF22] (see Theorem H in the Introduction).
Let us now move on to the last statement of the theorem and show that $u \equiv 0$ if $f(t) > 0$ for any $t > 0$. If not, then $u = u_0(x_N)$, where $u_0 : [0, +\infty) \to [0, , +\infty)$ is a bounded, positive and strictly increasing solution to the ODE $-u'' = f(u)$ on $(0, +\infty)$. By the first integral of the ode we have that also u' is bounded and, from this information, we immediately infer that $f(\sup u_0) = 0$. A contradiction. □

Proof of Theorem 3.1.3. As in the proof of Theorem 3.1.2, either $u \equiv 0$ (and we are done) or $u > 0$, $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$ and u is stable on $\mathbb{R}^N_+$. Also, from (3.1.11), we have

$$f(t) \geq At - B \qquad \text{for any } t > 0, \tag{3.2.6}$$

for some constants $A > 0$ and $B \geq 0$.
By Lemma 3.4.2 the profile of u at infinity defined by

$$\overline{u}(x') := \lim_{x_N \to +\infty} u(x', x_N), \qquad \text{for any } x' \in \mathbb{R}^{N-1},$$

is a non-negative member of $L^1_{loc}(\mathbb{R}^{N-1})$ and $f(\overline{u}) \in L^1_{loc}(\mathbb{R}^{N-1})$. Furthermore, the sequence of non-negative functions $(u_n)_{n \geq 1}$ defined by

$$u_n(x) := \tilde{u}(x', x_N + n), \qquad \text{for any } x \in \mathbb{R}^N,$$

converges to $\overline{u}$ in $L^1_{loc}(\mathbb{R}^N)$. Also observe that $u_n \in C^0(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N)$ for any $n \geq 1$.

Let us prove that $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1})$ and that $\overline{u}$ is a stable solution to $-\Delta \overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^{N-1})$.

Under the assumptions on f, there exists $t_0 > 0$ such that $f'(t) > 0$ on (t_0, ∞). Then

$$|f(u_n)| = f(u_n) = f(u_n)\mathbf{1}_{\{u_n \leq t_0\}} + f(u_n)\mathbf{1}_{\{u_n > t_0\}} \leq \|f\|_{L^\infty([0,t_0])} + f(\overline{u}) \in L^1_{loc}(\mathbb{R}^N),$$

and so

$$f(u_n) \longrightarrow f(\overline{u}) \qquad \text{in} \quad L^1_{loc}(\mathbb{R}^N), \tag{3.2.7}$$

by the dominated convergence theorem.

Since $-\Delta u_n = f(u_n)$ in $\mathbb{R}^N_+$ and $u_n \longrightarrow \overline{u}$ in $L^1_{loc}(\mathbb{R}^N)$, $f(u_n) \longrightarrow f(\overline{u})$ in $L^1_{loc}(\mathbb{R}^N)$, we get that $\overline{u}$ solves $-\Delta \overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^N)$. The latter implies that $-\Delta \overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^{N-1})$, since $\overline{u}$ depends only on the first $N - 1$ variables.

To prove that $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1})$ we recall that u is stable on $\mathbb{R}^N_+$. Therefore, Proposition 2.5 in [Cab+20] tell us that

$$\int_{B(z,R)} |\nabla u|^2 \leq C' R^{N-2} \Big(\frac{1}{R^N} \int_{B(z,2R)} u \Big)^2 \qquad \text{for any } B(z, 2R) \subset\subset \mathbb{R}^N_+, \tag{3.2.8}$$

where $C' > 0$ is a dimensional constant.

Set $e_N = (0, ..., 0, 1) \in \mathbb{R}^N$, pick any $r \geq R_0$, where R_0 is given by item (i) of Lemma 3.4.1 and consider any ball $B(z, r) \subset \mathbb{R}^N$. There exists an integer $\bar{n}$, depending only on z and r, such that $B(z + ne_N, 4r) \subset\subset \mathbb{R}^N_+$ for any $n \geq \bar{n}$. Hence, by (3.2.8) (with $R = r$) we deduce that

$$\int_{B(z,r)} |\nabla u_n|^2 = \int_{B(z+ne_N,r)} |\nabla u|^2 \leq C' r^{N-2} \Big(\frac{1}{r^N} \int_{B(z+ne_N,2r)} u \Big)^2 \qquad \text{for any } n \geq \bar{n}.$$

Now, for any $n \geq \bar{n}$, an application of Lemma 3.4.1 (with $R = 2r$) yields

$$\int_{B(z,r)} |\nabla u_n|^2 \leq C' r^{N-2} \Big(\frac{1}{r^N} \int_{B(z+ne_N,2r)} u \Big)^2 \leq C' r^{N-2} \Big(\frac{C_0 (2r)^N}{r^N} \Big)^2 = C_1(f, N) r^{N-2}, \tag{3.2.9}$$

where $C_1 = C_1(f, N) > 0$ is a constant (independent of n).

Moreover, if we denote by $[u_{z,n,r}]$ the average value of u over the open ball $B(z + ne_N, r)$ then, for any $n \geq \bar{n}$, the Poincaré inequality gives

$$\begin{aligned} \int_{B(z,r)} u_n^2 &= \int_{B(z+ne_N,r)} u^2 \leq 2 \int_{B(z+ne_N,r)} |u - [u_{z,n,r}]|^2 + 2 \int_{B(z+ne_N,r)} [u_{z,n,r}]^2 \\ &\leq C(N) r^2 \int_{B(z+ne_N,r)} |\nabla u|^2 + C(N) r^N \Big(\frac{1}{r^N} \int_{B(z+ne_N,r)} u \Big)^2 \\ &\leq C(N) r^2 \int_{B(z,r)} |\nabla u_n|^2 + C(N) r^N \Big(\frac{1}{r^N} \int_{B(z+ne_N,r)} u \Big)^2 \\ &\leq C(N) r^2 C_1 r^{N-2} + C(N) C_0(f, N)^2 r^N = C_2(f, N) r^N, \end{aligned} \tag{3.2.10}$$

where in the latter we have used (3.2.9), as well as Lemma 3.4.1 with $R = r$, since we also have that $B(z + ne_N, 2r) \subset\subset \mathbb{R}^N_+$.

By summarizing, we have proved that ,

$$\int_{B(z,r)} |\nabla u_n|^2 + u_n^2 \leq C_3(f, N, r) \qquad \text{for any } n \geq \bar{n}, \tag{3.2.11}$$

where $C_3 > 0$ is independent of n. Hence, the sequence (u_n) is bounded in $H^1_{loc}(\mathbb{R}^N)$ and so, we can find a subsequence, still denoted by (u_n), and $v \in H^1_{loc}(\mathbb{R}^N)$ such that

$$u_n \longrightarrow v \qquad \text{weakly in} \quad H^1_{loc}(\mathbb{R}^N). \tag{3.2.12}$$

The latter and the fact that $u_n \longrightarrow \overline{u}$ in $L^1_{loc}(\mathbb{R}^N)$ immediately imply that $\overline{u} = v \in H^1_{loc}(\mathbb{R}^N)$. Hence, $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1})$, since $\overline{u}$ is independent of the variable x_N.

Next we prove that $\overline{u}$ is a stable solution to $-\Delta \overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^{N-1})$. To this purpose, we observe that u_n is strictly increasing in $\mathbb{R}^N_+$, hence stable, therefore

$$\int f'(u_n) \varphi^2 \leq \int |\nabla \varphi|^2 \qquad \text{for any } \varphi \in C^\infty_c(\mathbb{R}^N_+), \tag{3.2.13}$$

and so

$$\int |f'(u_n)| \varphi^2 = \int f'(u_n) \varphi^2 + 2 \int [f'(u_n)]^- \varphi^2 \leq \int |\nabla \varphi|^2 + \int 2[f'(0)]^- \varphi^2 \tag{3.2.14}$$

by the convexity of f. Hence, by Fatou's Lemma, we have

$$\int |f'(\overline{u})|\varphi^2 \leq \int |\nabla\varphi|^2 + \int 2[f'(0)]^-\varphi^2 \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^N_+). \tag{3.2.15}$$

The latter immediately imply that $f'(\overline{u}) \in L^1_{loc}(\mathbb{R}^N_+)$, and so $f'(\overline{u})$ belongs to $L^1_{loc}(\mathbb{R}^{N-1})$ as well as to $L^1_{loc}(\mathbb{R}^N)$ (since $f'(\overline{u})$ only depends on the first $N-1$ variables). Also

$$\begin{aligned} |f'(u_n)| &= |f'(u_n)|\mathbf{1}_{\{u_n \leq t_0\}} + |f'(u_n)|\mathbf{1}_{\{u_n > t_0\}} \\ &\leq \|f'\|_{L^\infty([0,t_0])} + f'(u_n)\mathbf{1}_{\{u_n > t_0\}} \leq \|f'\|_{L^\infty([0,t_0])} + |f'(\overline{u})| \in L^1_{loc}(\mathbb{R}^N) \end{aligned} \tag{3.2.16}$$

and so, $f'(u_n) \longrightarrow f'(\overline{u})$ in $L^1_{loc}(\mathbb{R}^N)$ by the dominated convergence theorem. Then, passing to the limit in (3.2.13) we get

$$\int f'(\overline{u})\varphi^2 \leq \int |\nabla\varphi|^2 \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^N_+), \tag{3.2.17}$$

Now we consider a function $\eta \in C_c^\infty(\mathbb{R}_+)$ such that $supp\, \eta \subset (-1,1)$ and $\int_{\mathbb{R}} \eta^2 = 1$, and we insert in the stability inequality (3.2.17) the test function $\varphi_n(x', x_N) := \phi(x')\eta_n(x_N)$, where $\phi \in C_c^\infty(\mathbb{R}^{N-1})$ and $\eta_n = \frac{1}{n}\eta(\frac{x_N - 4n}{n})$. Note that this choice is possible since φ_n is supported in $\mathbb{R}^N_+$ by construction. Hence

$$\begin{aligned} &\int_{\mathbb{R}^N_+} f'(\overline{u})\phi^2\eta_n^2 \leq \int_{\mathbb{R}^N_+} |\nabla\phi|^2\eta_n^2 + \int_{\mathbb{R}^N_+} \phi^2(\eta_n')^2 \\ &(\int_{\mathbb{R}^{N-1}} f'(\overline{u})\phi^2)(\int_{\mathbb{R}_+} \eta_n^2) \leq (\int_{\mathbb{R}^{N-1}} |\nabla\phi|^2)(\int_{\mathbb{R}_+} \eta_n^2) + (\int_{\mathbb{R}^{N-1}} \phi^2)(\int_{\mathbb{R}_+} (\eta_n')^2) \\ &\int_{\mathbb{R}^{N-1}} f'(\overline{u})\phi^2 \leq \int_{\mathbb{R}^{N-1}} |\nabla\phi|^2 + (\int_{\mathbb{R}^{N-1}} \phi^2)(\int_{\mathbb{R}_+} (\eta_n')^2) \end{aligned} \tag{3.2.18}$$

and the last term in the latter tends to zero, when $n \to \infty$, since $\int_{\mathbb{R}_+} (\eta_n')^2 = \frac{1}{n^2}\int_{\mathbb{R}} (\eta')^2 \leq \frac{2\|\eta'\|^2_{L^\infty(\mathbb{R}_+)}}{n}$. Passing to the limit into (3.2.18) we immediately get the stability of $\overline{u}$.

By summarizing, we have proved that $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1})$ is a stable solution to $-\Delta\overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^{N-1})$. Since $N-1 \leq 9$, we can apply Theorem 5 in [DF23] to conclude that $\overline{u} \in C^2(\mathbb{R}^{N-1})$, i.e., $\overline{u}$ is a classical stable solution to $-\Delta\overline{u} = f(\overline{u})$ on $\mathbb{R}^{N-1}$. We can therefore invoke Theorem 1 (or Theorem 2) in [DF22] to prove that $\overline{u}$ is constant. This result and the monotonicity of u imply that u is bounded on $\mathbb{R}^N_+$. The desired conclusion then follows by applying Theorem 4 in [DF22] (see Theorem H in the Introduction).

Let us now move on to the last statement of the theorem and show that $u \equiv 0$ if $f(0) = 0$. If not, then $u = u_0(x_N)$, where $u_0 : [0, +\infty) \to (0, +\infty)$ is a bounded, positive and strictly increasing function satisfying the ODE $-u_0'' = f(u_0)$ on $[0, +\infty)$. Let us prove that this is impossible. Indeed, by the first integral of the ode we get that also u' is bounded, and so $f(\sup u_0) = 0$. Since f is non-negative and convex with $f(0) = 0$, f must be non-decreasing too, so $0 \leq f(u_0(t)) \leq f(\sup u_0) = 0$ for any $t \geq 0$. Therefore, $-u_0'' = 0$, $u_0 > 0$ on $(0, +\infty)$ and $u_0(0) = 0$, so that $u_0(t) = \alpha t$ with $\alpha > 0$, contradicting the boundedness of u_0. □

Proof of Theorem 3.1.4. Unless we add dummy variables, we may and do assume that $N = 10$. By the strong maximum principle, either $u \equiv 0$ (and we are done) or $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. Thus u is stable on $\mathbb{R}^N_+$ and so, Theorem 7.1 in [Cab+20] tell us that

$$\forall\, p > 1 \quad \exists C_1 = C_1(N,p,R) > 0 \quad : \quad \|u\|_{L^p(B(x,\frac{R}{4}))} \leq C_1 \|u\|_{L^1(B(x,R))} \quad \forall\ B(x,R) \subset\subset \mathbb{R}^N_+. \tag{3.2.19}$$

Then, using the second condition in (3.1.12) we get that

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_2 = C_2(N,p,R,f,\theta) > 0 \quad \text{such that} \\ &\|f(u)\|_{L^p(B(x,\frac{R}{4}))} \leq C_2 \Big[1 + \|u\|^{\theta}_{L^{p\theta}(B(x,\frac{R}{4}))} \Big] \qquad \forall\ B(x,R) \subset\subset \mathbb{R}^N_+, \end{aligned} \tag{3.2.20}$$

where $C_2 = C_2(N,p,R_0,f,\theta) > 0$ is a constant.
Now, the first condition in (3.1.12) and item (i) of Lemma 3.4.1 yield

$$\int_{B(x,R_0)} u \leq C_3 \qquad \forall\ B(x,2R_0) \subset\subset \mathbb{R}^N_+, \tag{3.2.21}$$

where $C_3 = C_3(N,f,R_0) > 0$ is a constant.
From (3.2.21) and (3.2.19)-(3.2.20) with $R = R_0$, we deduce that

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_4 = C_4(N,p,R_0,f,\theta) > 0 \quad \text{such that} \\ &\|u\|_{L^p(B(x,\frac{R_0}{4}))} + \|f(u)\|_{L^p(B(x,\frac{R_0}{4}))} \leq C_4 \qquad \forall\ B(x,2R_0) \subset\subset \mathbb{R}^N_+. \end{aligned} \tag{3.2.22}$$

Then, by standard interior elliptic estimates, we have

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_5 = C_5(N,p,R_0) > 0 \quad \text{such that} \\ &\|u\|_{W^{2,p}(B(x,\frac{R_0}{8}))} \leq C_5 \Big[\|u\|_{L^p(B(x,\frac{R_0}{4}))} + \|f(u)\|_{L^p(B(x,\frac{R_0}{4}))} \Big] \qquad \forall\ B(x,2R_0) \subset\subset \mathbb{R}^N_+. \end{aligned} \tag{3.2.23}$$

Now we choose any $p > \frac{N}{2}$ in the latter and we apply (3.2.22) and Sobolev embedding to get that

$$\|u\|_{L^\infty(B(x,\frac{R_0}{8}))} \leq C_5 \qquad \forall\ B(x,2R_0) \subset\subset \mathbb{R}^N_+, \tag{3.2.24}$$

where $C_5 = C_5(N,p,R_0,f,\theta) > 0$ is a constant.
From the latter we deduce that u is bounded on the half-space $\{x \in \mathbb{R}^N_+ : x_N > 4R_0\}$. The monotonicity of u then implies that u is bounded on $\mathbb{R}^N_+$. The desired conclusion the follows by invoking Theorem 4 in [DF22] (see Theorem H in the Introduction).

The last statement of the theorem follows exactly as in the last part of the proof of Theorem 3.1.2.

□

Proof of Corollary 3.1.1. By the strong maximum principle, either $u \equiv 0$ (and we are done) or $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. When $f(0) \geq 0$, any positive solution to (3.1.1), which is also bounded on finite strips, is strictly monotone in the x_N direction (see Theorem 3.1 of [Far20]). The desired conclusions then follow by applying Theorem 3.1.2 (resp. Theorem 3.1.3 or Theorem 3.1.4) □

Proof of Theorem 3.1.5. For any $R \geq 1$ and any $x \in \mathbb{R}^N_+$ set $u_R(x) := u(Rx)$ and $B_R = B(0,R)$. Then, u_R is a stable solution to

$$\begin{cases} -\Delta u_R = R^2 f(u_R) & \text{in} \quad \mathbb{R}^N_+, \\ \quad u_R \geq 0 & \text{in} \quad \mathbb{R}^N_+, \\ \quad u_R = 0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{3.2.25}$$

We can therefore apply Proposition 5.5 in [Cab+20] to (3.2.25) to infer that

$$\int_{B_{R/2}\cap\mathbb{R}^N_+} |\nabla u|^2 \leq C' R^{N-2} \Big(\frac{1}{R^N} \int_{B_R \cap \mathbb{R}^N_+} u \Big)^2, \tag{3.2.26}$$

where $C' > 0$ is a dimensional constant.

Now, since f is non-negative and $R \geq 1$, we observe that u_R also solves

$$-\Delta u_R = R^2 f(u_R) \geq f(u_R) \quad \text{in } \mathbb{R}^N_+,$$

thus, in view of the superlinearity condition (3.1.13) on f, we can apply Theorem 3 in [Sir21] to u_R with $\Omega = B_2 \cap \mathbb{R}^N_+$, $T = \partial\mathbb{R}^N_+ \cap B_2$, $\omega = \Omega' = B_1 \cap \mathbb{R}^N_+$, $A^1 = Id$, $\lambda = 1$, $q = 2N$, $\Lambda = \omega_N^{1/2N}$, $B = H = 0$, $\varepsilon = 1/2$, $p = 1$, to get

$$\int_{B_R\cap\mathbb{R}^N_+} u \leq CR^N, \qquad \forall\ R \geq 1,$$

where $C = C(N,f) > 0$ is a constant.

By plugging the latter into (3.2.27) and recalling that $N \leq 4$, we obtain

$$\int_{B_R\cap\mathbb{R}^N_+} |\nabla u|^2 \leq C'' R^2, \qquad \forall\ R \geq 1,$$

for some positive constant C''. Hence, according to Theorem 3.4.1, up to a rotation in the first $N-1$ variables, we have that $u(x) = u_0(x_{N-1}, x_N)$, where $u_0 \in C^2(\overline{\mathbb{R}^2_+})$ satisfies

$$\begin{cases} -\Delta u_0 = f(u_0) & \text{in} \quad \mathbb{R}^2_+, \\ \quad u_0 \geq 0 & \text{in} \quad \mathbb{R}^2_+, \\ \quad u_0 = 0 & \text{on} \quad \partial\mathbb{R}^2_+. \end{cases}$$

Since $f \not\equiv 0$ by (3.1.13) and $f(0) \geq 0$, Theorem 2.5 in [FS17] implies that $u_0 \equiv 0$, which in turn implies $u \equiv 0$ and $f(0) = 0$. Furthermore, by plugging $u = 0$ into the stability condition we immediately get that $f'(0) \leq 0$, and the desired conclusion $f'(0) = 0$ then follows from the monotonicity assumption on f. □

Proof of Theorem 3.1.6. By proceeding as in the proof of Theorem 3.1.5 we immediately get

$$\int_{B_{R/2}\cap\mathbb{R}^N_+} |\nabla u|^2 \leq C' R^{N-2}\Big(\frac{1}{R^N}\int_{B_R\cap\mathbb{R}^N_+} u\Big)^2, \qquad \forall\ R \geq 1, \tag{3.2.27}$$

where $C' > 0$ is a dimensional constant.
Next we use the growth assumption (3.1.14) to obtain that

$$\Big(\frac{1}{R^N}\int_{B_R\cap\mathbb{R}^N_+} u\Big)^2 \leq C(N)R^2 \ln R, \quad \text{for } R >> 1, \tag{3.2.28}$$

where $C(N) > 0$ is a dimensional constant.
The desired conclusion then follows by applying Theorem 3.4.1 of the present paper and Theorem 2.5 in [FS17] exactly as in the last part of the proof of Theorem 3.1.5 (recall that $f \not\equiv 0$ by assumption). □

Proof of Theorem 3.1.7. By the strong maximum principle, either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^N_+$. The desired conclusion will follow if we rule out the latter case. For contradiction, assume that $u > 0$ in $\mathbb{R}^N_+$. Since f is a non-negative, convex function with $f(0) = 0$, the function $t \mapsto \frac{f(t)}{t}$ is non-negative and non-decreasing on $(0, +\infty)$, so $\lim_{t\to+\infty} \frac{f(t)}{t} \in [0, +\infty]$. Moreover, since $f \not\equiv 0$, we necessarily have $\lim_{t\to+\infty} \frac{f(t)}{t} > 0$. Next we distinguish two cases:
Case 1 : $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$.

By assumption, there exist a point $\bar{x} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^N_+$ and an open ball $B(\bar{x}, \bar{R}) \subset\subset \mathbb{R}^N_+$ such that u is stable in $\mathbb{R}^N_+ \setminus \overline{B(\bar{x}, \bar{R})}$. If we set $\bar{c} = \frac{\bar{x}_N - \bar{R}}{4} > 0$, then :

(i) for any $x \in \{x \in \mathbb{R}^N_+ : x_N > \bar{c}\} \setminus \overline{B(\bar{x}, \bar{R} + \bar{c})}$ we have that $B(x, \bar{c}) \subset\subset \mathbb{R}^N_+$ and that $B(x, \frac{\bar{c}}{2}) \subset \mathbb{R}^N_+ \setminus \overline{B(\bar{x}, \bar{R})}$ and therefore u is stable in $B(x, \frac{\bar{c}}{2})$

and

(ii) u is stable in the strip $\{x \in \mathbb{R}^N_+ : x_N < 2\bar{c}\}$.

When we are in the situation described in item (i), an application of Theorem 1.2 in [Cab+20] provides

$$\|u\|_{L^\infty(B(x,\frac{\bar{c}}{4}))} \leq C_1 \|u\|_{L^1(B(x,\frac{\bar{c}}{2}))}, \tag{3.2.29}$$

where $C_1 = C_1(N, \bar{c}) > 0$ is a constant.
Since f satisfies $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$ and $B(x, \bar{c}) \subset\subset \mathbb{R}^N_+$, we can apply item (ii) of Lemma 3.4.1 to get that

$$\int_{B(x,\frac{\bar{c}}{2})} u \leq C_2, \tag{3.2.30}$$

where $C_2 = C_2(f, N, \bar{c})$ is a positive constant.
From (3.2.29) and (3.2.30) we obtain that $u \leq C_1C_2$ on $\{x \in \mathbb{R}^N_+ : x_N \geq \bar{c}\} \setminus \overline{(B(\bar{x}, \bar{R} + \bar{c})}$

and so, by the continuity of u on $\overline{B(\bar{x}, \bar{R}+\bar{c})}$, we deduce that u is bounded in the closed half-space $\{x \in \mathbb{R}^N_+ : x_N \geq \bar{c}\}$.

On the other hand, for any $x \in \{x \in \mathbb{R}^N_+ : x_N < \bar{c}\}$, we have that $x \in B((x',0),\bar{c}) \cap \mathbb{R}^N_+$. Since u is stable in $\{x \in \mathbb{R}^N_+ : x_N < 2\bar{c}\}$ (see item (ii) above), we can apply Theorem 6.1 in [Cab+20] to obtain that

$$\|u\|_{L^\infty(B((x',0),\bar{c})\cap\mathbb{R}^N_+)} \leq C_3 \|u\|_{L^1(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)}, \qquad \forall\, x' \in \partial\mathbb{R}^N_+, \tag{3.2.31}$$

where $C_3 = C_3(N,\bar{c}) > 0$ is a constant.
Since f satisfies $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$, we can apply Theorem 3 in [Sir21] to get that

$$\int_{B((x',0),2\bar{c})\cap\mathbb{R}^N_+} u \leq C_4, \qquad \forall\, x' \in \partial\mathbb{R}^N_+, \tag{3.2.32}$$

where $C_4 = C_4(f,N,\bar{c})$ is a positive constant.
From (3.2.31) and (3.2.32) we infer that u is bounded on the strip $\omega := \{x \in \mathbb{R}^N_+ : x_N < \bar{c}\}$ and therefore u is bounded on $\overline{\mathbb{R}^N_+}$. Since $f(0) \geq 0$ and $u > 0$ is bounded, Theorem 3.1 in [Far20] ensures that u is strictly increasing in the x_N-direction. Therefore, u is stable in $\mathbb{R}^N_+$ and we can apply once again Theorem 6.1 in [Cab+20] to obtain that, for any $R > 1$ and for any $x, y \in B(0, \frac{R}{2}) \cap \mathbb{R}^N_+$,

$$|u(x) - u(y)| \leq C_4 \|u\|_{L^\infty(\mathbb{R}^N_+)} \frac{|x-y|^\alpha}{R^\alpha},$$

where $C_4 > 0$ is a dimensional constant.
By letting $R \to +\infty$, we obtain that $u(x) = u(y)$ for any $x, y \in \mathbb{R}^N_+$, which entails $u \equiv 0$ in $\mathbb{R}^N_+$, since $u = 0$ on $\partial\mathbb{R}^N_+$. A contradiction with $u > 0$ in $\mathbb{R}^N_+$.

Case 2 : $\lim_{t\to+\infty} \frac{f(t)}{t} \in (0,+\infty)$.

In this case it is easily seen that f is globally Lipschitz-continuous and Corollary 1.3 in [BCN97a] implies that u is strictly increasing in the x_N-direction. Hence, u is stable on $\mathbb{R}^N_+$ and so Theorem 1.2 in [Cab+20] tell us that

$$\|u\|_{L^\infty(B(x,\frac{R_0}{2}))} \leq C\|u\|_{L^1(B(x,R_0))} \qquad \text{for any} \quad B(x,R_0) \subset\subset \mathbb{R}^N_+, \tag{3.2.33}$$

where $C = C(N,R_0) > 0$ is constant.
From the latter and item (i) of Lemma 3.4.1 we deduce that u is bounded on the half-space $\{x \in \mathbb{R}^N_+ : x_N > 2R_0\}$, therefore, the monotonicity of u then implies that u is bounded on $\mathbb{R}^N_+$. We can then proceed as in the previous case to achieve a contradiction.

So far, we have proved that $u \equiv 0$ on $\mathbb{R}^N_+$, the identities $f(0) = f'(0) = 0$ then follow as in the proof of Theorem 3.1.5. □

Proof of Theorem 3.1.8. By the strong maximum principle, either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^N_+$. The desired conclusion will follow if we rule out the latter case. For contradiction, assume that $u > 0$ in $\mathbb{R}^N_+$.

By assumption, there exist a point $\bar{x} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^N_+$ and an open ball $B(\bar{x}, \bar{R}) \subset\subset \mathbb{R}^N_+$ such that u is stable in $\mathbb{R}^N_+ \setminus \overline{B(\bar{x}, \bar{R})}$. If we set $\bar{c} = \frac{\bar{x}_N - \bar{R}}{16} > 0$, then :

(i) for any $x \in \{x \in \mathbb{R}^N_+ : x_N > \bar{c}\} \setminus \overline{B(\bar{x}, \bar{R} + \bar{c})}$ we have that $B(x, \bar{c}) \subset\subset \mathbb{R}^N_+$ and that $B(x, \frac{\bar{c}}{2}) \subset \mathbb{R}^N_+ \setminus \overline{B(\bar{x}, \bar{R})}$ and therefore u is stable in $B(x, \frac{\bar{c}}{2})$

and

(ii) u is stable in the strip $\{x \in \mathbb{R}^N_+ : x_N < 8\bar{c}\}$.

Now we observe that, without loss of generality, we can restrict ourselves to the 10-dimensional case. Indeed, if u is stable in an open set $\omega \subset \mathbb{R}^N_+$, $N \leq 9$, then the function $\tilde{u}(y, x) := u(x)$ is stable in the open cylinder $\mathbb{R}^{10-N} \times \omega \subset \mathbb{R}^{10}_+$.
Hence, from now on, we set $N = 10$.
When we are in the situation described in item (i), an application of Theorem 7.1 in [Cab+20] provides

$$\forall\, p > 1 \quad \exists C_1 = C_1(N, p, \bar{c}) > 0 \quad : \quad \|u\|_{L^p(B(x, \frac{\bar{c}}{8}))} \leq C_1 \|u\|_{L^1(B(x, \frac{\bar{c}}{2}))} \quad \forall\; B(x, \bar{c}) \subset\subset \mathbb{R}^N_+. \tag{3.2.34}$$

Then, using the second condition in (3.1.15) we get that

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_2 = C_2(N, p, \bar{c}, f, \theta) > 0 \quad \text{such that} \\ &\|f(u)\|_{L^p(B(x, \frac{\bar{c}}{8}))} \leq C_2 \Big[1 + \|u\|^{\theta}_{L^{p\theta}(B(x, \frac{\bar{c}}{8}))}\Big] \qquad \forall\; B(x, \bar{c}) \subset\subset \mathbb{R}^N_+. \end{aligned} \tag{3.2.35}$$

Since f satisfies $\lim\limits_{s \to +\infty} \frac{f(s)}{s} = +\infty$ and $B(x, \bar{c}) \subset\subset \mathbb{R}^N_+$, we can apply item (ii) of Lemma 3.4.1 to get that

$$\int_{B(x, \frac{\bar{c}}{2})} u \leq C_3, \tag{3.2.36}$$

where $C_3 = C_3(f, N, \bar{c})$ is a positive constant.
From (3.2.34)-(3.2.36) we deduce that

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_4 = C_4(N, p, \bar{c}, f, \theta) > 0 \quad \text{such that} \\ &\qquad \|u\|_{L^p(B(x, \frac{\bar{c}}{8}))} + \|f(u)\|_{L^p(B(x, \frac{\bar{c}}{8}))} \leq C_4 \qquad \forall\; B(x, \bar{c}) \subset\subset \mathbb{R}^N_+. \end{aligned} \tag{3.2.37}$$

Then, by standard interior elliptic estimates, we have

$$\begin{aligned} &\forall\, p > 1 \quad \exists C_5 = C_5(N, p, \bar{c}) > 0 \quad \text{such that} \\ &\|u\|_{W^{2,p}(B(x, \frac{\bar{c}}{16}))} \leq C_5 \Big[\|u\|_{L^p(B(x, \frac{\bar{c}}{8}))} + \|f(u)\|_{L^p(B(x, \frac{\bar{c}}{8}))}\Big] \qquad \forall\; B(x, \bar{c}) \subset\subset \mathbb{R}^N_+. \end{aligned} \tag{3.2.38}$$

Now we choose any $p > \frac{N}{2}$ in the latter and we apply (3.2.37) and Sobolev embedding to get that

$$\|u\|_{L^\infty(B(x, \frac{\bar{c}}{16}))} \leq C_5 \qquad \forall\; B(x, \bar{c}) \subset\subset \mathbb{R}^N_+, \tag{3.2.39}$$

where $C_5 = C_5(N, p, \bar{c}, f, \theta) > 0$ is a constant.
From the latter we see that u is bounded on $\{x \in \mathbb{R}^N_+ : x_N \geq \bar{c}\} \setminus \overline{(B(\bar{x}, \bar{R} + \bar{c})}$ and so, by

the continuity of u on $\overline{B(\bar{x}, \bar{R}+\bar{c})}$, we deduce that u is bounded in the closed half-space $\{x \in \mathbb{R}^N_+ : x_N \geq \bar{c}\}$.

On the other hand, for any $x \in \{x \in \mathbb{R}^N_+ : x_N < \bar{c}\}$, we have that $x \in B((x',0),\bar{c}) \cap \mathbb{R}^N_+$. Then we proceed as in the first part of the proof, but now we use boundary estimates. Since u is stable in $\{x \in \mathbb{R}^N_+ : x_N < 8\bar{c}\}$ (see item (ii) above), we can apply Theorem 7.2 in [Cab+20] to obtain that

$$\begin{aligned}&\forall\, p>1 \quad \exists C_6 = C_6(N,p,\bar{c}) > 0 \quad : \\ &\|u\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)} \leq C_6 \|u\|_{L^1(B((x',0),8\bar{c})\cap\mathbb{R}^N_+)} \qquad \forall\, x' \in \partial\mathbb{R}^N_+\end{aligned} \tag{3.2.40}$$

and using the second condition in (3.1.15) as before, we infer that

$$\begin{aligned}&\forall\, p>1 \quad \exists C_7 = C_7(N,p,\bar{c},f,\theta) > 0 \quad \text{such that} \\ &\|f(u)\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)} \leq C_7\Big[1 + \|u\|^{\theta}_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)}\Big] \qquad \forall\, x' \in \partial\mathbb{R}^N_+.\end{aligned} \tag{3.2.41}$$

Moreover, since f satisfies $\lim\limits_{s\to+\infty} \frac{f(s)}{s} = +\infty$, we can apply Theorem 3 in [Sir21] to get that

$$\int_{B((x',0),8\bar{c})\cap\mathbb{R}^N_+} u \leq C_7, \qquad \forall\, x' \in \partial\mathbb{R}^N_+, \tag{3.2.42}$$

where $C_7 = C_7(f,N,\bar{c})$ is a positive constant.
From (3.2.40)-(3.2.42) we deduce that

$$\begin{aligned}&\forall\, p>1 \quad \exists C_8 = C_8(N,p,\bar{c},f,\theta) > 0 \quad \text{such that} \\ &\|u\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)} + \|f(u)\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+)} \leq C_8 \qquad \forall\, x' \in \partial\mathbb{R}^N_+.\end{aligned} \tag{3.2.43}$$

Then, by standard elliptic boundary estimates on half-balls, we have

$$\begin{aligned}&\forall\, p>1 \quad \exists C_9 = C_9(N,p,\bar{c}) > 0 \quad \text{such that} \\ &\|u\|_{W^{2,p}(B((x',0),\bar{c})\cap\mathbb{R}^N_+)} \leq C_9\Big[\|u\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+))} + \|f(u)\|_{L^p(B((x',0),2\bar{c})\cap\mathbb{R}^N_+))}\Big] \qquad \forall\, x' \in \partial\mathbb{R}^N_+.\end{aligned} \tag{3.2.44}$$

Now we choose any $p > \frac{N}{2}$ in the latter and we apply (3.2.43) and Sobolev embedding to get that

$$\|u\|_{L^\infty(B((x',0),\bar{c})\cap\mathbb{R}^N_+)} \leq C_{10} \qquad \forall\, x' \in \partial\mathbb{R}^N_+, \tag{3.2.45}$$

where $C_{10} = C_{10}(N,p,\bar{c},f,\theta) > 0$ is a constant.

From the latter we deduce that u is bounded on the the strip $\{x \in \mathbb{R}^N_+ : x_N < \bar{c}\}$ and therefore u is bounded on $\mathbb{R}^N_+$. Then, Theorem 4 in [DF22] (see Theorem H in the Introduction) implies that $u = u_0(x_N)$, where $u_0 : [0,+\infty) \to [0,+\infty)$ is a bounded, positive and strictly increasing function satisfying the ODE $-u_0'' = f(u_0)$ on $[0,+\infty)$. Let us prove that this is impossible. Indeed, by the first integral of the ode we get that also u' is bounded, so that $f(\sup u_0) = 0$ and $0 \leq f(u_0(t)) \leq f(\sup u_0) = 0$ for any $t \geq 0$. Therefore, $-u_0'' = 0$, $u_0 > 0$ on $(0,+\infty)$ and $u_0(0) = 0$, so that $u_0(t) = \alpha t$ with $\alpha > 0$, contradicting the boundedness of u_0.

So far, we have proved that $u \equiv 0$ on $\mathbb{R}^N_+$, the identities $f(0) = f'(0) = 0$ then follow as in the proof of Theorem 3.1.5. $\square$

3.3 Proofs of Theorems 3.1.9-3.1.12

Proof of Theorem 3.1.9. By Lemma 3.4.2 we know that $\overline{u} \in L^1_{loc}(\mathbb{R}^{N-1})$ and so $v = -\overline{u}$ belongs to $L^1_{loc}(\mathbb{R}^{N-1})$ and satisfies

$$\begin{aligned} -\Delta v \leq -f(\overline{u}) \leq 0 \qquad &\text{in} \quad \mathcal{D}'(\mathbb{R}^{N-1}), \\ v \leq 0 \qquad &\text{on} \quad \mathbb{R}^{N-1}. \end{aligned} \tag{3.3.1}$$

Since $N-1 = 1, 2$, then $\overline{u}$ must be constant (see for instance Theorem 3.2.18 in [DV24] or Theorem 3.2.24 in [Hör94]). Therefore, $f(\overline{u}) = 0$ and so $\overline{u} = 0$. The monotonicity of u, then implies $u \equiv 0$ on $\mathbb{R}^N_+$. □

Proof of Theorem 3.1.10. For $N = 2, 3$ the conclusion follows from Theorem 3.1.9. When $N \geq 4$, let us distinguish the case $p = 1$ from the case $p > 1$. When $p = 1$, by the strong maximum principle either $u \equiv 0$, and we are done, or $u > 0$ on $\mathbb{R}^N_+$. Let us prove that the latter case does not occur. Indeed, if $u > 0$, then by multiplying the differential inequality $-\Delta u \geq u$ by $\frac{\varphi^2}{u}$, where $\varphi \in C^1_c(\mathbb{R}^N_+)$, and integrating by parts we get $\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^N_+} \varphi^2$. The latter and the variational characterization of λ_1 (the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition) tell us that $\lambda_1(B) \geq 1$ for any open ball $B \subset\subset \mathbb{R}^N_+$. This is clearly impossible, since $\lambda_1(B)$ approaches zero when B is chosen arbitrary large.

When $p > 1$, Lemma 3.4.2 tells us that $\overline{u} \in L^p_{loc}(\mathbb{R}^{N-1})$ and

$$-\Delta \overline{u} \geq \overline{u}^p \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^{N-1}). \tag{3.3.2}$$

Since $1 < p \leq \frac{N-1}{N-3} = \frac{(N-1)}{(N-1)-2}$, we can apply Theorem 2.1 in [MP01] to obtain that $\overline{u}$ is a constant (necessarily equal to zero). The monotonicity of u, then implies $u \equiv 0$ on $\mathbb{R}^N_+$. □

Proof of Theorem 3.1.11. By the strong maximum principle, either $u \equiv 0$, and we are done, or $u > 0$. Again, by applying the strong maximum principle to $\frac{\partial u}{\partial x_N}$ we see that, either $\frac{\partial u}{\partial x_N} = 0$, and we are done, or $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. The desired conclusion will follow if we rule out the latter case. For contradiction, assume that $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. The latter implies that u is stable on $\mathbb{R}^N_+$ and so Theorem 1.2 in [Cab+20] tell us that

$$\|u\|_{L^\infty(B(x,\frac{R_0}{2}))} \leq C\|u\|_{L^1(B(x,R_0))} \qquad \text{for any} \quad B(x, R_0) \subset\subset \mathbb{R}^N_+, \tag{3.3.3}$$

where $C = C(N, R_0) > 0$ is constant.
The assumption (3.1.20) and (3.3.3) allow us to apply item (i) of Lemma 3.4.1 to deduce that u is bounded on the half-space $\{x \in \mathbb{R}^N_+ : x_N > 2R_0\}$. The monotonicity of u then implies that u is bounded on $\mathbb{R}^N_+$. As a consequence, we see that $\overline{u}$ is a bounded classical solution to $-\Delta \overline{u} = f(\overline{u})$ in $\mathbb{R}^{N-1}$, $N - 1 \leq 8$. We can therefore invoke Theorem 1 (or Theorem 2) in [DF22] to prove that $\overline{u}$ is constant. Therefore, $f(\overline{u}) = 0$ and so $\overline{u} \equiv 0$. The monotonicity of u, then implies $u \equiv 0$ on $\mathbb{R}^N_+$. A contradiction.

If we further assume (3.1.12), then we can proceed as in the proof of Theorem 3.1.4 to get, once again, that u is bounded on some half-space $\{x \in \mathbb{R}^N_+ : x_N > c > 0\}$. Then, we conclude exactly as in the previous case (note that $N-1 \leq 9$). $\square$

Proof of Theorem 3.1.12. Let us first prove that either $u \equiv 0$ or u depends only on the first $N-1$ variables. To this aim, as in the proof of Theorem 3.1.11, it is enough to rule out the case : $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$. For contradiction, assume $u > 0$ and $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$, then u is stable on $\mathbb{R}^N_+$ and so by Lemma 3.4.2 we have $\overline{u} \in L^p_{loc}(\mathbb{R}^{N-1})$ and $|\overline{u} - u_n|^p \leq |\overline{u}|^p$ a.e. on $\mathbb{R}^N$. Hence, by the dominated convergence theorem we infer that $u_n \longrightarrow \overline{u}$ in $L^p_{loc}(\mathbb{R}^N)$ and $\overline{u}$ solves $-\Delta \overline{u} = \overline{u}^p$ in $\mathcal{D}'(\mathbb{R}^{N-1})$.

Next we prove that $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1}) \cap L^{p+1}_{loc}(\mathbb{R}^N)$ and that $\overline{u}$ is a stable solution to $-\Delta \overline{u} = f(\overline{u})$ in $\mathcal{D}'(\mathbb{R}^{N-1})$.

To this end, we recall that u is stable on $\mathbb{R}^N_+$ and so, from Proposition 4 in [Far07] (applied with $\gamma = 1$) we get

$$\int_{B(z,R)} |\nabla u|^2 + |u|^{p+1} \leq C' R^{N-2\frac{p+1}{p-1}} \qquad \text{for any } B(z, 2R) \subset\subset \mathbb{R}^N_+, \tag{3.3.4}$$

where $C' > 0$ is a constant depending only on N and p.

Set $e_N = (0, ..., 0, 1) \in \mathbb{R}^N$ and pick any open ball $B(z, r) \subset \mathbb{R}^N$. Then, there exists an integer $\bar{n}$, depending only on z and r, such that $B(z + ne_N, 2r) \subset\subset \mathbb{R}^N_+$ for any $n \geq \bar{n}$. Hence, by (3.3.4) we deduce that

$$\int_{B(z,r)} |\nabla u_n|^2 + |u_n|^{p+1} = \int_{B(z+ne_N,r)} |\nabla u|^2 + |u|^{p+1} \leq C' r^{N-2\frac{p+1}{p-1}} \qquad \text{for any } n \geq \bar{n}.$$

From the latter, and by a standard diagonal argument, we can find a subsequence, still denoted by (u_n), and $v \in H^1_{loc}(\mathbb{R}^N) \cap L^{p+1}_{loc}(\mathbb{R}^N)$ such that

$$\begin{aligned} u_n &\longrightarrow v \qquad \text{weakly in} \quad H^1_{loc}(\mathbb{R}^N), \\ u_n &\longrightarrow v \qquad \text{weakly in} \quad L^{p+1}_{loc}(\mathbb{R}^N). \end{aligned} \tag{3.3.5}$$

Since we already know that $u_n \longrightarrow \overline{u}$ in $L^1_{loc}(\mathbb{R}^N)$, from (3.3.5) we immediately get that $\overline{u} = v \in H^1_{loc}(\mathbb{R}^N) \cap L^{p+1}_{loc}(\mathbb{R}^N)$. Hence, $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1}) \cap L^{p+1}_{loc}(\mathbb{R}^{N-1})$, since $\overline{u}$ is independent of the variable x_N. The stability of $\overline{u}$ then follows exactly as in the last part of the proof of Theorem 3.1.3.

By summarizing, we have proved that $\overline{u} \in H^1_{loc}(\mathbb{R}^{N-1}) \cap L^{p+1}_{loc}(\mathbb{R}^N)$ is a stable solution to $-\Delta \overline{u} = \overline{u}^p$ in $\mathcal{D}'(\mathbb{R}^{N-1})$, then Theorem 1.1 in [DDF11] yields that $\overline{u} \in C^2(\mathbb{R}^{N-1})$. To conclude, we apply Theorem 1 in [Far07] to get that $\overline{u} = 0$. The monotonicity of u, then implies $u \equiv 0$ on $\mathbb{R}^N_+$. A contradiction.

So far, we proved that either $u \equiv 0$ or u depends only on the first $N-1$ variables. To complete the proof we need only to prove the claim in item (ii). To this end, we recall that all smooth positive solutions to the critical equation $-\Delta v = v^{\frac{K+2}{K-2}}$ in $\mathbb{R}^K$, with $K \geq 3$, have been classified in [CGS89] and are given by

$$v(x) = (a + b|x - x_0|^2)^{-\frac{K-2}{2}} \qquad x \in \mathbb{R}^K, \tag{3.3.6}$$

for some $a, b > 0$ with $1 = abK(K-2)$ and $x_0 \in \mathbb{R}^K$. $\square$

3.4 Auxiliary results

In this section we state and prove some auxiliary results. They will be used to prove the main results of this article, but they are also interesting in their own right. Let us begin with two preliminaries results concerning non-negative solutions to the differential inequalities in the half-space. The first result is part of folklore, and we state it in a form useful for our purposes, while the second one deals with monotone solutions.

Lemma 3.4.1. *Assume $N \geq 2$ and let $u \in C^2(\mathbb{R}^N_+)$ be a solution to*

$$\begin{cases} -\Delta u \geq f(u) & \text{in } \mathbb{R}^N_+, \\ u \geq 0 & \text{in } \mathbb{R}^N_+, \end{cases} \tag{3.4.1}$$

where $f \in C^0([0, +\infty))$.

(i) If f satisfies

$$f(t) \geq At - B \qquad \text{for any } t > 0, \tag{3.4.2}$$

where $A > 0$ and $B \geq 0$, then, there exist $R_0 = R_0(A, N) > 0$ and $C_0 = C_0(A, B, N) > 0$ such that for any $R \geq R_0$ and any open ball $B(z, 2R) \subset\subset \mathbb{R}^N_+$ we have

$$\int_{B(z,R)} u \leq C_0 R^N. \tag{3.4.3}$$

(ii) If f satisfies

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty, \tag{3.4.4}$$

then, for any $R > 0$ and any open ball $B(z, 2R) \subset\subset \mathbb{R}^N_+$ we have

$$\int_{B(z,R)} u \leq C_1, \tag{3.4.5}$$

where $C_1 = C_1(N, f, R) > 0$ is a constant.

Proof. (i) Denote by ϕ_1 the positive first eigenfunction of $-\Delta$ on the open unit ball $B(0, 1)$, such that $\max_{B(0,1)} \phi_1 = 1$ and by $\lambda_1 > 0$ the corresponding first eigenvalue. For any ball $B(z, 2R) \subset\subset \mathbb{R}^N_+$ set $\phi_{R,z}(x) := \phi_1(\frac{x-z}{2R})$, then

$$\begin{cases} -\Delta\phi_{R,z} = \frac{\lambda_1}{4R^2}\phi_{R,z} & \text{in } B(z, 2R), \\ \phi_{R,z} > 0 & \text{in } B(z, 2R), \\ \phi_{R,z} = 0 & \text{on } \partial B(z, 2R). \end{cases}$$

From (3.4.1) and (3.4.2) we deduce that

$$\int_{B(z,2R)} (Au - B)\phi_{R,z} \leq -\int_{B(z,2R)} \Delta u\phi_{R,z} \leq -\int_{B(z,2R)} u\Delta\phi_{R,z} = \frac{\lambda_1}{4R^2}\int_{B(z,2R)} u\phi_{R,z}. \tag{3.4.6}$$

Set $R_0 = \sqrt{\frac{\lambda_1}{2A}} > 0$, then for any $R \geq R_0$ we get

$$\int_{B(z,2R)} Au\phi_{R,z} \leq \frac{\lambda_1}{4R_0^2}\int_{B(z,2R)} u\phi_{R,z} + B\int_{B(z,2R)} \phi_{R,z} = \frac{A}{2}\int_{B(z,2R)} u\phi_{R,z} + B\int_{B(z,2R)} \phi_{R,z}, \tag{3.4.7}$$

so that

$$\frac{2B}{A}\int_{B(z,2R)} \phi_{R,z} \geq \int_{B(z,2R)} u\phi_{R,z} \geq \inf_{B(z,R)} \phi_{R,z}\int_{B(z,R)} u = \inf_{B(0,\frac{1}{2})} \phi_1 \int_{B(z,R)} u. \tag{3.4.8}$$

From the latter we get

$$\int_{B(z,R)} u \leq (\inf_{B(0,\frac{1}{2})} \phi_1)^{-1}\frac{2B}{A}\int_{B(z,2R)} \phi_{R,z} \leq (\inf_{B(0,\frac{1}{2})} \phi_1)^{-1}\frac{2B}{A}\int_{B(z,2R)} 1, \tag{3.4.9}$$

which implies (3.4.3).

(ii) For any $R > 0$ such that $B(z,2R) \subset\subset \mathbb{R}^N_+$, we set $A = \frac{\lambda_1}{4R^2} + 1$, then by (3.4.4) there exists $B = B(f,A) > 0$ such that $f(t) \geq At - B$ for any $t > 0$. We can therefore proceed as in the previous step to get the estimates (3.4.6) which, in turns, gives

$$\int_{B(z,2R)} u\phi_{R,z} \leq \int_{B(z,2R)} B\phi_{R,z} \tag{3.4.10}$$

and thus,

$$\int_{B(z,R)} u \leq (\inf_{B(0,\frac{1}{2})} \phi_1)^{-1} B\int_{B(z,2R)} \phi_{R,z} \leq (\inf_{B(0,\frac{1}{2})} \phi_1)^{-1} B\int_{B(z,2R)} 1. \tag{3.4.11}$$

The latter implies the desired conclusion. □

Lemma 3.4.2. *Assume $N \geq 2$. Let $u \in C^2(\mathbb{R}^N_+)$ be a solution of*

$$\begin{cases} -\Delta u \geq f(u) & in \quad \mathbb{R}^N_+, \\ u \geq 0 & in \quad \mathbb{R}^N_+, \\ \frac{\partial u}{\partial x_N} \geq 0 & in \quad \mathbb{R}^N_+, \end{cases} \tag{3.4.12}$$

where $f \in C^0([0,+\infty))$ satisfies

$$f(t) \geq At - B \qquad \text{for any } t > 0, \tag{3.4.13}$$

where $A > 0$ and $B \geq 0$. Then,

(i) $\overline{u}(x') := \lim_{x_N \to +\infty} u(x', x_N) \in [0,+\infty]$ for any $x' \in \mathbb{R}^{N-1}$ and $\overline{u} : \mathbb{R}^{N-1} \mapsto [0,+\infty]$ belongs to $L^1_{loc}(\mathbb{R}^{N-1})$.

(ii) The sequence of functions $(u_n)_{n\geq 1}$ defined by

$$u_n(x) := \tilde{u}(x', x_N + n), \qquad \text{for any } x \in \mathbb{R}^N,$$

converges to $\overline{u}$ *in* $L^1_{loc}(\mathbb{R}^N)$.[7]

(*iii*) $f(\overline{u}) \in L^1_{loc}(\mathbb{R}^{N-1})$ *and* $\overline{u}$ *solves*

$$-\Delta\overline{u} \geq f(\overline{u}) \qquad in \quad \mathcal{D}^{'}(\mathbb{R}^{N-1}). \tag{3.4.14}$$

Proof. We observe that $\tilde{u}$ is measurable and so, $(u_n)_{n\geq 1}$ is a sequence of measurable functions satisfying

$$0 \leq u_n \leq u_{n+1} \leq \overline{u} \qquad \text{on } \mathbb{R}^N, \qquad \text{for any } n \geq 1. \tag{3.4.15}$$

Set $e_N = (0, ..., 0, 1) \in \mathbb{R}^N$, pick any $r \geq R_0$, where R_0 is given by item (i) of Lemma 3.4.1 and consider any open ball $B(z, r) \subset \mathbb{R}^N$. There exists an integer $\bar{n}$, depending only on z and r, such that $B(z + ne_N, 2r) \subset\subset \mathbb{R}^N_+$ for any $n \geq \bar{n}$. Hence, by Lemma 3.4.1 we deduce that

$$\int_{B(z+ne_N,r)} u \leq C_0(f, N)r^N \qquad \text{for any } n \geq \bar{n},$$

and

$$\int_{B(z,r)} u_n = \int_{B(z+ne_N,r)} u \leq C_0(f, N)r^N \qquad \text{for any } n \geq \bar{n}.$$

The latter and (3.4.15) then imply that $(u_n)_{n\geq 1}$ is a well-defined sequence in $L^1_{loc}(\mathbb{R}^N)$ satisfying

$$\int_{B(z,r)} u_n \leq C_0(f, N)r^N \qquad \text{for any } n \geq 1.$$

Then, the monotone convergence theorem implies that $u_n \longrightarrow \overline{u}$ in $L^1_{loc}(\mathbb{R}^N)$. This proves items (i) and (ii).

Now, for any $n \geq 1$ and any $\varphi \in C^\infty_c(\mathbb{R}^N), \varphi \geq 0$, we have $0 \leq (f(u_n) + B)\varphi$, hence

$$\int (f(\overline{u}) + B)\varphi \leq \liminf_{n\to\infty} \int (f(u_n) + B)\varphi = \liminf_{n\to\infty} \int f(u_n)\varphi + \int B\varphi \tag{3.4.16}$$

by Fatou's Lemma. From the latter we deduce that $f(\overline{u}) \in L^1_{loc}(\mathbb{R}^N)$ (and so also in $L^1_{loc}(\mathbb{R}^{N-1})$, since it depends only on the first $N - 1$ variables) and

$$\int f(\overline{u})\varphi \leq \liminf_{n\to\infty} \int f(u_n)\varphi. \tag{3.4.17}$$

As above, pick an open ball $B(z, r)$ such that *supp* $\varphi \subset B(z, r)$, then we have $-\Delta u_n \geq f(u_n)$ in $B(z, r)$, for any $n \geq \bar{n}$. This property and (3.4.17) 1ead to

$$\begin{aligned} \int f(\overline{u})\varphi &\leq \liminf_{n\to\infty} \int f(u_n)\varphi \leq \liminf_{n\to\infty} - \int \Delta u_n\varphi = \liminf_{n\to\infty} - \int u_n\Delta\varphi \\ &= \lim_{n\to\infty} - \int u_n\Delta\varphi = - \int \overline{u}\Delta\varphi. \end{aligned} \tag{3.4.18}$$

The latter tell us that $-\Delta\overline{u} \geq f(\overline{u})$ in $\mathcal{D}^{'}(\mathbb{R}^N)$ and the desired conclusion then follows by recalling that $\overline{u}$ only depends on the first $N - 1$ variables. □

[7] Here, as usual, we have denoted by $\tilde{u}$ the extension of u with the value 0 outside $\mathbb{R}^N_+$. Also note that, the same convergence result holds true if we replace u_n by $u_n(x) := \tilde{u}(x', x_N + a_n)$, where $(a_n)_{n\geq 1}$ is any sequence of positive real numbers converging monotonically to $+\infty$.

The following theorem extends a result by N. Dancer valid for bounded stable solutions (see item (i) of Theorem 3 in [Dan05]). We remove the boundedness assumption and also make a slightly weaker hypotheses on the energy growth of the solutions over large balls.

Theorem 3.4.1. *Let $N \geq 2$ and $u \in C^2(\overline{\mathbb{R}^N_+})$ be a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{in} \quad \partial\mathbb{R}^N_+, \end{cases}$$

where $f \in C^1(\mathbb{R})$. Assume that

$$\int_{B(0,R)\cap\mathbb{R}^N_+} |\nabla u|^2 = O(R^2 \ln R) \quad \text{as } R \to +\infty. \tag{3.4.19}$$

Then, either

(i) u is a function depending only on x_N,

or,

(ii) (after a rotation in the first $N-1$ coordinates) u is a function depending only on (x_{N-1}, x_N) and strictly monotone in the x_{N-1} direction.

The proof of Theorem 3.4.1 makes use of the following proposition and of the subsequent corollary.

Proposition 3.4.1. *Assume $N \geq 2$. Let $V \in C^0(\overline{\mathbb{R}^N_+})$, $\overline{\lambda} \geq 0$ and $v \in C^1(\overline{\mathbb{R}^N_+})$ be a weak solution to*

$$\begin{cases} -\Delta v = Vv + \overline{\lambda} v & \text{in} \quad \mathbb{R}^N_+, \\ v > 0 & \text{in} \quad \mathbb{R}^N_+, \\ v = 0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{3.4.20}$$

Let $w \in C^1(\overline{\mathbb{R}^N_+})$ be a weak solution to

$$\begin{cases} -\Delta w = Vw & \text{in} \quad \mathbb{R}^N_+, \\ w \leq 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{3.4.21}$$

such that

$$\int_{B(0,R)\cap\mathbb{R}^N_+} (w^+)^2 = O(R^2 \ln R) \qquad \text{as } R \to +\infty. \tag{3.4.22}$$

Then,

$$\frac{w^+}{v} \equiv \text{constant} \quad \text{in } \mathbb{R}^N_+,$$

where $w^+ = \max(w, 0)$.

Corollary 3.4.1. *Let v, w, V and $\overline{\lambda}$ as in the statement of Theorem 3.4.1. Assume in addition that $w = 0$ on $\partial\mathbb{R}^N_+$. Then*

$$\frac{w}{v} = \text{constant} \quad \text{in } \mathbb{R}^N_+.$$

Let us now prove the results stated above.

Proof of Theorem 3.4.1. For any $R > 0$ set $B_R = B(0, R)$. As in the proof of Theorem 3 in [Dan05], we prove the existence of a positive function $v \in C^2(\overline{\mathbb{R}^N_+})$ such that

$$\begin{cases} -\Delta v = f'(u)v + \overline{\lambda} v & \text{in} \quad \mathbb{R}^N_+, \\ \qquad v = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

with $\overline{\lambda} = \lim\limits_{R\to+\infty} \lambda_R \geq 0$ where (λ_R, ϕ_R) is a pair eigenvalue-eigenfunction of $-\Delta\phi_R - f'(u)\phi_R = \lambda_R\phi_R$ in $B_R \cap \mathbb{R}^N_+$. Note that, since f' and u are continuous, then $f'(u)$ is bounded on compact sets of $\overline{\mathbb{R}^N_+}$ and Harnack inequality up to the boundary (Theorem 1.4 in [BCN96]) ensures that $(\phi_R)_{R\geq R_0}$ is uniformly bounded on $B_{R_0} \cap \mathbb{R}^N_+$ for any $R_0 > 0$.

Now, for $i \in \{1, \cdots, N-1\}$ we set $w = \frac{\partial u}{\partial x_i}$, then $w \in C^1(\overline{\mathbb{R}^N_+})$ is a weak solution to

$$\begin{cases} -\Delta w = f'(u)w & \text{in} \quad \mathbb{R}^N_+, \\ \qquad w = 0 & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

and, since (3.4.19) is in force, we get

$$\int_{B_R\cap\mathbb{R}^N_+} w^2 \leq \int_{B_R\cap\mathbb{R}^N_+} |\nabla w|^2 = O(R^2 \ln R) \qquad \text{as } R \to +\infty.$$

Hence, according to Corollary 3.4.1 we have

$$\frac{\partial u}{\partial x_i} = C_i v \quad \text{in } \mathbb{R}^N_+, \qquad \text{for any } i \in \{1, \cdots, N-1\},$$

for some constants C_i.

By an orthogonal rotation of axis in $\mathbb{R}^{N-1}$ we may and do suppose that $C_i = 0$ for any $i \in \{1, N-2\}$.

Now, either $C_{N-1} = 0$, and u is a function of x_N only, or $C_{N-1} \neq 0$ and, since $v > 0$, then $\frac{\partial u}{\partial x_{N-1}} = C_{N-1}v > 0$ if $C_{N-1} > 0$ (< 0 if $C_{N-1} < 0$). □

Proof of Theorem 3.4.1. Set $\sigma := \frac{w}{v} \in C^1(\mathbb{R}^N_+)$. Since v and w satisfy (3.4.20) and (3.4.21), then σ is a weak solution to $div(v^2\nabla\sigma) = \overline{\lambda}v^2\sigma$ in $\mathbb{R}^N_+$. Then, by interior elliptic regularity theory, we have that $\sigma \in H^2_{\text{loc}}(\mathbb{R}^N_+)$ and

$$div(v^2\nabla\sigma) = \overline{\lambda}v^2\sigma \quad \text{a.e. in } \mathbb{R}^N_+. \tag{3.4.23}$$

For any $\Psi \in C^{0,1}_c(\mathbb{R}^N)$, let B an open ball in $\mathbb{R}^{N-1}$ and $T > 0$ such that $\text{supp}(\Psi) \subset B \times (-T, T) := \mathcal{C}$. For any integer $m > \frac{1}{T}$, multiply (3.4.23) by $\sigma^+\Psi^2$ and integrate by parts over $\Omega_m \cap \mathcal{C}$, where $\Omega_m := \{(x', x_N) \in \mathbb{R}^N : x_N > \frac{1}{m}\}$, to obtain

$$-\int_{\Omega_m} v^2(\nabla\sigma.\nabla(\sigma^+\Psi^2)) - \underbrace{\int_{\mathcal{C}\cap\partial\Omega_m} \sigma^+\Psi^2\Big(v\frac{\partial w}{\partial x_N} - w\frac{\partial v}{\partial x_N}\Big)\, d\mathcal{H}^{N-1}}_{I_m} = \int_{\Omega_m} \overline{\lambda}(w^+)^2\Psi^2 \geq 0. \tag{3.4.24}$$

Next we prove that $\lim_{m\to\infty} I_m = 0$. To this end we observe that,

$$
\begin{aligned}
I_m &= \int_{\mathcal{C}\cap\partial\Omega_m} \sigma^+\Psi^2 v \frac{\partial w}{\partial x_N} d\mathcal{H}^{N-1} - \int_{\mathcal{C}\cap\partial\Omega_m} \sigma^+\Psi^2 w \frac{\partial v}{\partial x_N} d\mathcal{H}^{N-1} \\
&= \underbrace{\int_{\mathcal{C}\cap\partial\Omega_m} w^+\Psi^2 \frac{\partial w}{\partial x_N} d\mathcal{H}^{N-1}}_{I_{1,m}} - \underbrace{\int_{\mathcal{C}\cap\partial\Omega_m} \sigma^+\Psi^2 w^+ \frac{\partial v}{\partial x_N} d\mathcal{H}^{N-1}}_{I_{2,m}}.
\end{aligned}
$$

Let us first deal with the term $I_{1,m}$. In this case we have

$$
I_{1,m} = \int_B \left(w^+\Psi^2 \frac{\partial w}{\partial x_N} \right)(x', 1/m)dx' := \int_B g_m(x')dx'
$$

with $|g_m| \le \|w\|^2_{C^1(\overline{\mathbb{R}_+\cap\mathcal{C}})} \|\Psi\|^2_{C^0(\overline{\mathbb{R}^N_+\cap\mathcal{C}})} \mathbf{1}_B \in L^1(B)$ and $\lim\limits_{m\to+\infty} g_m(x') = 0$ for any $x' \in B$. Thus, the dominated convergence theorem implies that

$$
\lim_{m\to+\infty} I_{1,m} = 0. \tag{3.4.25}
$$

Now, we focus on the term $I_{2,m}$. Since σ^+ is not defined up to the boundary, we can not proceed directly as above. Nevertheless, by Hopf's Lemma we have $\frac{\partial v}{\partial x_N} > 0$ on $\partial\mathbb{R}^N_+$ and so, by a standard compactness argument there exists $\varepsilon > 0$ such that

$$
\frac{\partial v}{\partial x_N}(x) \ge \varepsilon \quad \text{in } \overline{\mathcal{C} \cap \{(x', x_n) \in \mathbb{R}^N_+ : x_N < \varepsilon\}}, \tag{3.4.26}
$$

which, in turn, implies that

$$
v(x) \ge \varepsilon x_N \quad \text{in } \overline{\mathcal{C} \cap \{(x', x_n) \in \mathbb{R}^N_+ : x_N < \varepsilon\}}. \tag{3.4.27}
$$

Also, since $w^+ = 0$ on $\partial\mathbb{R}^N_+$ we have

$$
w^+(x) \le \|w\|_{C^1(\overline{\mathbb{R}_+\cap\mathcal{C}})} x_N \quad \text{in } \overline{\mathcal{C} \cap \{(x', x_n) \in \mathbb{R}^N_+ : x_N < \varepsilon\}}. \tag{3.4.28}
$$

Now we estimate the term $I_{2,m}$ for $m > \max\{\frac{1}{\varepsilon}, \frac{1}{T}\}$. Since

$$
I_{2,m} = \int_B \left(\sigma^+\Psi^2 w^+ \frac{\partial v}{\partial x_N} \right)(x', 1/m)dx' := \int_B h_m(x')dx'
$$

and

$$
\sigma^+(x', 1/m) = \frac{w^+(x', 1/m)}{v(x', 1/m)} \le \frac{\|w\|_{C^1(\overline{\mathbb{R}_+\cap\mathcal{C}})}}{m} \frac{m}{\varepsilon} = \frac{\|w\|_{C^1(\overline{\mathbb{R}_+\cap\mathcal{C}})}}{\varepsilon} \qquad \text{for any } x' \in B,
$$

where in the latter inequality we have used (3.4.27)-(3.4.28), we deduce that h_m is bounded on B, independently on $m > \max\{\frac{1}{\varepsilon}, \frac{1}{T}\}$. Furthermore, $\lim\limits_{m\to+\infty} h_m(x') = 0$ for any $x' \in B$, therefore $\lim\limits_{m\to+\infty} I_{2,m} = 0$ by the dominated convergence theorem. Hence, $\lim\limits_{m\to+\infty} I_m = 0$.

We also observe that $v^2(\nabla\sigma.\nabla(\sigma^+\Psi^2))$ is bounded on $\mathbb{R}^N_+$ and zero a.e. outside $B \times (0,T)$, hence $\int_{\Omega_m} v^2(\nabla\sigma.\nabla(\sigma^+\Psi^2)) \longrightarrow \int_{\mathbb{R}^N_+} v^2(\nabla\sigma.\nabla(\sigma^+\Psi^2))$, as $m \to +\infty$.

Therefore, by letting $m \to +\infty$ in (3.4.24), we get that

$$-\int_{\mathbb{R}^N_+} v^2(\nabla\sigma.\nabla(\sigma^+\Psi^2)) \geq 0. \tag{3.4.29}$$

Then, by Young inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi^2 \leq -2\int_{\mathbb{R}^N_+} v^2\sigma^+\Psi(\nabla\sigma^+.\nabla\Psi) &\leq \int_{\mathbb{R}^N_+}(v\Psi|\nabla\sigma^+|)(2v\sigma^+|\nabla\Psi|)\\ &\leq \frac{1}{2}\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi^2 + 2\int_{\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi|^2,\end{aligned}$$

which entails

$$\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi^2 \leq 4\int_{\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi|^2. \tag{3.4.30}$$

For $R > 1$, let Ψ_R be defined by

$$\Psi_R(x) := \mathbf{1}_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R}\mathbf{1}_{B_R\setminus B_{\sqrt{R}}}(x),$$

then

$$\int_{B_R\cap\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi_R|^2 \leq \frac{4}{\ln R^2}\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} \frac{(w^+)^2}{|x|^2}. \tag{3.4.31}$$

Now we use the identity $\frac{1}{|x|^2} = \int_{|x|}^R \frac{2}{t^3} + \frac{1}{R^2}$ and the assumption (3.4.22) to get

$$\begin{aligned}\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} \frac{(w^+)^2}{|x|^2} &= \int_{\sqrt{R}}^R\int_{(B_t\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} \frac{2(w^+)^2}{t^3}dxdt + \int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} \frac{(w^+)^2}{R^2}\\ &\leq 2C\ln^2 R + C\ln R.\end{aligned} \tag{3.4.32}$$

where $C > 0$ is a constant independent of R (such that $\int_{B_R\cap\mathbb{R}^N_+}(w^+)^2 \leq CR^2\ln R$). From the latter, (3.4.30) and (3.4.31) we deduce that

$$\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi_R^2 \leq 4\int_{\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi_R|^2 \leq 48C \qquad \text{for } R >> 1, \tag{3.4.33}$$

In particular, by letting $R \to \infty$, we get $\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2 \leq 48C$, that is $v^2|\nabla\sigma^+|^2 \in L^1(\mathbb{R}^N_+)$. Now, from (3.4.29), (3.4.33) and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}&\int_{B_R\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi_R^2 \leq -2\int_{\mathbb{R}^N_+} v^2\sigma^+\Psi_R(\nabla\sigma^+.\nabla\Psi_R)\\ &\leq 2\Big(\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi_R^2\Big)^{1/2}\Big(\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi_R|^2\Big)^{1/2}\\ &\leq 2\Big(\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi_R^2\Big)^{1/2}\Big(\int_{B_R\cap\mathbb{R}^N_+} v^2(\sigma^+)^2|\nabla\Psi_R|^2\Big)^{1/2}\\ &\leq 8\sqrt{C}\Big(\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Psi_R^2\Big)^{1/2} \leq 8\sqrt{C}\Big(\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2\Big)^{1/2}\end{aligned}$$

Finally, since $v^2|\nabla\sigma^+|^2 \in L^1(\mathbb{R}^N_+)$, we deduce from the dominated convergence theorem that $\int_{(B_R\setminus B_{\sqrt{R}})\cap\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2 \to 0$ as $R \to +\infty$, and so $\int_{\mathbb{R}^N_+} v^2|\nabla\sigma^+|^2 = 0$. Since $v > 0$ in $\mathbb{R}^N_+$, we infer that $\nabla\sigma^+ = 0$ almost everywhere in $\mathbb{R}^N_+$ and so $\frac{w^+}{v} = \sigma^+ = \textit{constant}$ in $\mathbb{R}^N_+$. $\square$

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Propriétés qualitatives et aspects géométriques de solutions d'équations aux dérivées partielles non linéaires dans des domaines non bornés

Résumé.

Cette thèse porte sur l'étude qualitative et la classification des solutions de l'équation de Poisson semi-linéaire posée sur des domaines non bornés comme, le demi-espace ou, plus généralement, des épigraphes. Nous avons mis en évidence des résultats de monotonie en toute dimension et pour des non-linéarités localement ou globalement Lipschitzienne. Ces résultats reposent sur l'utilisation de nouveaux principes de comparaison et sur la célèbre méthode des hyperplans mobiles (introduite par James Serrin) que l'on a adapté à la géométrie des domaines considérés. Ces nouveaux résultats nous ont permis de démontrer la conjecture concernant le problème sur-déterminé de Serrin dans des épigraphes réguliers et minorés. Enfin, nous avons classifié les solutions éventuellement non bornées du problème de Dirichlet homogène de l'équation de Poisson dans le demi-espace.

Mots clés : Propriétés qualitatives et classification des solutions, Equations aux Dérivées Partielles (EDP), régularité, comportement à l'infini.

Qualitative properties and geometric aspects of solutions to nonlinear partial differential equations in unbounded domains

Abstract.

This thesis focuses on the qualitative study and classification of solutions to the semi-linear Poisson equation posed on unbounded domains such as half-space or, more generally, epigraphs. We have highlighted monotonicity results in all dimensions and for locally or globally Lipschitz-continuous nonlinearities. These results are based on the use of new comparison principles and the famous moving palne method (introduced by James Serrin) which we have adapted to the geometry of considered domains. These new results allowed us to prove the conjecture concerning Serrin's overdetermined problem in smooth epigraphs bounded from below. Finally, we classified possibly unbounded solutions of the homogeneous Dirichlet problem of the Poisson equation in half-space.

Keywords: Qualitative properties and classification of solutions, Partial differential Equations (PDE), regularity, infinite behavior.